Competition for informal and formal hiring

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Abstract

Hiring an employee from an internal recommendation is an established informal recruitment method in many labour markets as it helps to recruit high ability candidates. The conditions under which a worker is hired by a referral are not well established though. For an individual worker, the probability of being hired by a referral is heterogeneous. I propose a model that accounts for the heterogeneity in this probability due to competition for this valuable recruitment method. Firms pay a cost to acquire a recruitment technology that allows them to access this informal channel. I explore two cases of informal hiring. In the first case there is no correlation between the referees ability and their ability to refer a high quality candidate. In the second case there is a positive correlation. Firms hire by either a referral or in an open competitive market. The referral is advantageous to the firm as it provides information on the productivity of the potential new employee. I show that for an unemployed worker the probability they are hired by a referral is related to the cost of referral, the proportion of high ability candidates and the signal quality of the recruitment technology.

1 Introduction

The use of referrals to hire workers has been found to be be quite extensive across different industries, jobs and countries (Ioannides and Datcher Loury (2004), Marmaros and Sacerdote (2002), Burks et al. (2015), Pellizzari (2010) and Antoninis (2006)). However, the prevalence of referrals is heterogeneous, varying for instance by country (Pellizzari (2010)) or the education requirements for a job opening (Brown et al. (2016)). The literature as far as I know has not fully explored what conditions might be driving this heterogeneity. I argue that competition amongst firms for access to referrals is a potential mechanism to help understand what is driving this heterogeneity.

I consider referrals to be some form of informal hiring and hiring in the competitive market as a formal method. For a referral an incumbent employee recommends a potential hire to their firm. I explore two different reasons for why the incumbent employee makes a referral. In the first case the firm acquires some costly technology that allows their incumbent employee to identify a workers ability. This can be a signal about the potential employees ability. In this model this equivalent to identifying before a worker has had a job how productive they will be. The specifics of the technology are not modeled here. However, the main point is that incumbent employees ability and so productivity has no bearing on how good they are at using this technology to identify highly productive workers. Therefore, I refer to this case as type-independent. The benchmark the technology is measured against for identifying highly productive workers is the competitive market or the formal method. The incumbent worker has access to the technology and this means that relative to the competitive market they are able to more accurately identify unemployed workers from their social network who are highly productive.

In the second case the firm again acquires some costly technology that allows their incumbent employee to identify a potential workers ability. This can also be a signal about the potential employees ability. However, now there is some correlation between the incumbent employees productivity and how good they are at identifying highly productive workers. Those workers who themselves are highly productive are more likely to identify other highly productive workers. This assumption can be interpreted as homophily. Where, homophily is the tendency for people to share links with people of similar characteristics such as race, age, background etc. In this paper I will focus on homophily of productivity. That is highly productive workers are more likely to have social links with other highly productive workers. The idea is that firms use their current employees social network to find these high ability potential employees (Montgomery (1991)). When a firm is looking to hire a new employee if they simply hire from the formal market they rely on their own ability to assess the quality of a candidate and so there is uncertainty as to the true ability of the worker. However, by using the information they have on their own employees ability e.g. by observing them work. They can infer the ability of an employee directly referred by their current employee, reducing the uncertainty of the potential employees true ability.

An implication of this model is that if firms only use their referrals from their high ability workers, then referred workers are of higher ability than non-referred ones. Hence, referrals are an important method to hire high ability workers and so improve the firms performance. Indeed Montgomery (1991) shows that having an employee likely to connect with high ability unemployed workers creates a valuable "option" for the firm to exercise in their future hiring decisions. This "option" is increasing in the level of homophily due to the increased likelihood of connecting to high ability workers. Casella and Hanaki (2008) extend this model and allow unemployed workers to signal their ability to firms in the formal market at some cost. They show that referrals are still used to hire workers even in situations where the workers signaling perfectly reveals their type. This framework suggest that referrals are beneficial to both firms and workers in the labour market. Firms are able to reduce the uncertainty of new employees abilities and workers do not have to spend effort acquiring human capital to signal their type. Hence, referrals should be a large source of filling vacancies at firms. However, the empirical data shows that of the various methods to hire a worker (direct applications, recruitment firms, adverts etc.) there is a large amount of heterogeneity. For example Pellizzari (2010) shows that in Europe the probability a worker was hired from a referral at firms with less than twenty employees is 45%. Whereas, for firms with more than one hundred employees this drops to 19%. Whilst, this heterogeneity has been exploited to answer interesting economic questions (Galenianos (2014)). There has been less work done on what are the mechanisms behind the reason for this heterogeneity.

I propose that firms are able to access a referral at some cost. The value of the referral, since it gives them a hiring advantage relative to the market through the signal they get, attracts firms to compete for access to them. However, by increasing the number of firms acquiring access to referrals the increased competition means firms must increase their wages to attract the high ability workers. This central trade-off, that the less firms who use referrals the more valuable it is and so the more firms are attracted to try and access them, is not present in the previous work cited. Once competition for access to referrals is introduced then, whilst referrals may still be present in a particular firm or industry, the prevalence of them will vary depending on the appropriate parameters chosen. For example I show that if the signal of a referral is increasing. So that a firm is more certain of the potential productivity of a new worker and if the cost of access to a referral is low enough. Then the probability that a unemployed worker is hired by a referral will increase then decrease. However, if the cost of access to a referral is sufficiently high. Then the probability is only increasing. The reason for this difference is due to the level of competition. When the cost is low and the signal is marginally better than hiring in the competitive market. This induces a lot of firms to access the informal hiring method, a referral. This increases the probability a worker is hired by a referral, as more firms are using the method. With the level of competition so high due to the low cost, many non productive workers are being hired by a referral as firms take a chance that they will have a link to an productive worker. As the signal quality improves, the level of uncertainty about the potential employees productivity reduces, so firms a start to be less willing to incur the cost of a referral. They are less willing to take the chance they get an unproductive worker. This decreases the probability of being hired by a referral. Essentially, the improved quality of the signal does not benefit the firm as the driver of its use is the low cost. On the other hand, if the cost is high and the signal quality is low (marginally better than hiring in the competitive market). Then the level of competition in referrals is low. But any marginal improvement in the signal quality will be valuable information to a firm as there is not much competition. This then increases the probability of being hired by a referral.

Section 2 reviews the related literature and the contribution of this paper. Section 3 details the model and assumptions made. Section 4 and 5 outline two different cases of the model. I conclude in section 6. The details of the proofs are in appendix A.

2 Related Literature

As mentioned earlier Montgomery (1991) and Casella and Hanaki (2008) have models of referrals. Where, the firm has some access to information on the potential new hire. However, the firms exogenously get access to referrals and so unlike my model can say nothing about the how firms optimally respond, in terms of the decision to acquire access to referrals, to changes in the economy. For example, to an increase in the proportion of highly productive workers. Galenianos (2014) incorporates referrals to transmit information about potential workers into a standard labour search model [1]. Galenianos (2014) studies the implication of referrals on heterogeneity in industry aggregate matching efficiency. This paper does not have heterogeneous workers and so can not demonstrate the role that heterogeneity has on the prevalence of referrals as this paper does. Pellizzari (2010) also incorporates referrals into a labour search model but assumes that the probability of finding a job through a referral is fixed. Whereas, I am interested in how this probability can vary.

There are other arguments put forward for why referrals may be used by firms as a hiring strategy. One in particular is that referrals are used as a monitoring mechanism. Heath (2018) has a model where referrals are used to alleviate a moral hazard problem, where a worker hired without a referral may slack on the job. By using a referral a firm can "monitor" a worker and induce high effort in the recipient of such a referral. However, this model requires an environment in a low wage setting where the proposer of the referral earns more than the recipient, who is must be earning close to the minimum wage. If the recipient is earning the minimum wage, then the firm is unable to punish the recipient worker directly for slacking on the job. Yet, the firm can do so indirectly by punishing the proposer [2]. This model only applies to minimum wage settings and so is not necessarily applicable to a more general setting.

3 Model

I initially set up the model for a finite number of workers and firms. However, the analysis later on will be studied in the limit as the number of workers and firms goes to infinity.

3.1 Workers

There are 2 periods, which can also be thought of as sequential markets. In period 1 there is a large number of individual workers born. In period 2 there are two overlapping generations of workers. Those newly born in this period and those who were born in the previous one. The workers are of two types, there are high (H) types who are productive and low (L) types who are unproductive. When they are born they are initially unemployed and seeking employment. Workers are observationally equivalent, but if they are hired their type is revealed to them and the hiring firm when they go to produce. They work at the end of the period in which they are hired. An H type worker produces 1 unit of output and nothing if they are an L type. As the workers do not know their own type before they work, they will accept the highest wage offered to them. Let $2N$ scale of the overall population size. Let the fraction λ_H be H types and so $\lambda_L = (1 - \lambda_H)$ are L types. Hence, there are $\lambda_H 2N$ H types and $(1 - \lambda_H)2N$ L types.

3.2 Firms

There is a large number of firms, aN[3] who are looking to hire a single worker. In period 1 they are have an unfilled vacancy and are looking to fill it. However, in period 2 if the firm hired in period 1 then the firm has an incumbent employee about to retire and they are looking to replace them. A firms profit is equal to the productivity of the hired employee minus the wage paid. The product price is exogenously given and normalized to unity. Firms set the wage before learning the productivity of the new employee.

3.3 Hiring

In period 1 a firm pays a cost κ to enter and hire a worker. They simply select a worker at random. This means that they can potentially hire a worker of either type. To simplify the analysis the firm pays the worker their expected productivity, which is λ_H since only H types are productive. In period 2 there are two methods for firms to hire a new worker, either they can use a formal or an informal method. Only the firms who entered in period 1 and paid the cost κ can use the informal hiring method. In general let there by μN firms who are able to make an informal offer to an unemployed worker. Here N scales the overall population size and the parameter $\mu \in \mathbb{R}_+$ represents the quantity of informal offers. In this paper formal refers to any hiring done in a competitive market e.g. fee entry condition[4]. Workers

[1] In particular Pissarides (1985)

[2] Furthermore, the proposer and recipient have to interact in close proximity to each other in order for the proposer to extract the docked income from the recipients poor performance. Heath (2018) tests this model empirically in a garment factory setting in Bangladesh. Where workers often live in an extended family residential compounds, known as a *bari*.

[3] As I am not studying the implications for unemployment it is assumed that there are more firms than workers $a > 2$.

[4] Any firm who did not pay κ to hire in period 1 can still hire a worker in period 2 but only via the competitive market.

are selected uniformly at random with no replacement, this ensures that no worker gets selected by two different firms.

Informal hiring is considered a referral from an incumbent employee to the firm about a potential new employee. The incumbent employee is able to only refer a single unemployed worker (see section 3.4 for more details). A referral is a recommendation from the incumbent employee to the firm to hire a particular worker. They have access to a signal about the workers ability. The incumbent relays this to the firm, hence they gain some informational advantage in recruitment over the formal (competitive) method. There are two cases of the model analysed in this paper. How the signal is interpreted will be different in each case. Hence, I will go into more detail about the signal in the following description of the cases.

The first case is referred to as Type-Independent. There is no correlation between the type of worker hired in period 1 and the signal they receive about the worker in period 2. The new worker is hired by a referral but there is a recruitment technology that a firm invests in, which allows the incumbent to receive a signal about a potential workers type independent of their own type[5]. The signal is the probability α that the unemployed employee connected to the firm is an H type. To fix ideas, if a firm were to simply select a worker at random the probability of selecting an H type is equal to the underlying distribution λ_H. Then, the signal α is informative in the sense that if the firm receives the signal α, this means that the probability the firms' incumbent worker has a link to an H type is greater than a random selection from the pool of unemployed workers. That is $\alpha > \lambda_H$. Since there are only two types the probability $1 - \alpha$ tells the firm the probability they have a connection with an L type. There is no assumption that the technology gives the firm a greater probability of connecting to an H type than an L type. This means $\alpha \in (0, 1)$ and it is possible to have $\alpha < 1 - \alpha$. In fact I will show later that in the equilibrium for a sufficiently low cost κ all that matters is that the informal method gives a greater probability of connecting to an H type α than the underlying distribution λ_H.

The second case is referred to as Type-Dependent. Now there *is* some correlation between the type of worker hired in period 1 and the signal they receive about the worker in period 2. Here, informal hiring is still thought of as a referral. However the main difference is that employed (incumbent) and unemployed workers are part of a social network and that this network exhibits homophily. That is people with similar characteristics tend to have social links. Homophily can be along many dimensions, but in this paper I will focus on homophily in productivity. Here, the interpretation is that between generations workers with the same productivity are more likely share a connection. In this particular setup this means that the probability an incumbent and unemployed worker of the same type share a social link is $\hat{\alpha}$ and it is less likely that they have a social link to different type. This means $\hat{\alpha} > 1 - \hat{\alpha}$ and this now can in thought of as an in-breeding bias. [6]. Furthermore, given the assumption of homophily in the network conditional on the type of worker the probability of connecting to the same type is greater than a random selection $\hat{\alpha} > \lambda_H$. Additionally homophily means the probability of connecting to the opposite type is less than a random selection. So if a firm has an incumbent L type their probability of connecting to an H type is $1 - \hat{\alpha}$. Hence, it is required that $\lambda_H > 1 - \hat{\alpha}$. This means that in this case the restriction $\hat{\alpha} > \lambda_H > 1 - \hat{\alpha}$ is imposed.

3.4 Network Structure

The opportunity for using informal hiring can be thought of as the social links formed between an incumbent employee and the unemployed workers. The process for forming these links is stochastic. The firm with a referral opportunity can form a single link to an unemployed worker of either type. If there is no correlation between the incumbent type and unemployed type. Then a firm with an incumbent employee, independent of their own type, has link is to an H type worker with probability α and an L type with probability $1 - \alpha$. A specific worker is chosen uniformly at random. However, if there is a correlation. Then conditional on the type hired in period 1, with probability $\hat{\alpha}$ the link is to the same type worker and with probability $1 - \hat{\alpha}$ it is to the opposite type of worker. Again the specific young worker is chosen uniformly at random from those of the appropriate type. Due to the stochastic nature of the network formation, some young may have many links and some may have none. This process generates a network of links characterized by the parameter α in the case of Type-Independence and $\hat{\alpha}$

[5]For example they may be sent to a conference to network or they are trained on how to assess people's potential productiveness even if they themselves are not of the same ability.

[6]Hence, the signal $\hat{\alpha}$ is now dependent on the type of the incumbent employee. So if they have an L type they are more likely to have a link to an unemployed L type. Whereas, previously it was independent and simply told their firm their probability of connection to an H type.

in the case of Type-Dependence [7].

3.5 Timing

The timing of the model is as follows.

- Firm i if they desire pays cost κ and hires an H or L worker.
- Firm i if they desire sets a referral offer w_{Ri}.
- Social links formed for μN firms.
- Incumbent worker relays this offer w_{Ri} to their unemployed connection.
- Unemployed worker j compares their offer(s) and either accepts one or rejects all.
- Workers who rejected or have no offers go on formal market which clears at competitive wage w_M.
- Production occurs.

4 Type-Independent Equilibrium

In this section if a firm pays a cost κ to enter the period 1 market to hire a worker in order to access the informal hiring method. Then the signal they receive in period 2 about a potential unemployed worker is independent of the type of worker they had hired. The equilibrium concept used is that of a perfect Bayesian equilibrium. The model will be solved using backwards induction, starting with the period 2 market wage.

4.1 The market wage

Since not all workers are hired from a referral, as they either receive no offers or reject their offer, there will be a pool of workers who can be hired in the formal market. They will be offered a wage equal to their expected productivity.

Given the technology an H type worker will receive a wage offer via a referral from an firm with probability $\frac{\alpha}{\lambda_H 2N}$. Similarly, an L type worker will receive a referral offer with probability $\frac{1-\alpha}{\lambda_L 2N}$. Hence, the probability that an H and an L type worker does not receive a *single* referral offer is given by

$$P\left(\text{no referral}|H\right) = \left(1 - \frac{\alpha}{\lambda_H 2N}\right)^{\mu N} \text{ and } P\left(\text{no referral}|L\right) = \left(1 - \frac{1-\alpha}{(1-\lambda_H)2N}\right)^{\mu N}$$

Moving from a finite number of firms to an infinite and using the following result that

$$N \to \infty \Rightarrow (1 - \frac{a}{N})^N \to \exp(-a)$$

Then as $N \to \infty$ the resulting probabilities are

$$P\left(\text{no ref.}|H\right) \to \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) \text{ and } P\left(\text{no ref.}|L\right) \to \exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)$$

Since only H type workers are productive, the expected productivity will be the output of an H type multiplied by the probability, conditional on not receiving a referral, that a worker is an H type. Applying Bayes' rule, the market wage is given by

$$\begin{aligned} w_M &\equiv P\left(H|\text{no ref}\right) \\ &= \frac{P\left(\text{no ref.}|H\right) P(H)}{P\left(\text{no ref.}|H\right) P(H) + P\left(\text{no ref.}|L\right)\left(1 - P(H)\right)} \end{aligned}$$

[7] In Montgomery (1991) there is some uncertainty over whether the firm is even able to make an offer. Hence, a firm can make use of a referral offer with probability τ. This paper will focus on firms who already have such a link and so can make use of a referral. This means that the focus of the analysis in this paper will be on the mass of referral opportunities μ. However, including link formation uncertainty via τ can be easily accommodated. To do this suppose there are q employment opportunities and each employment relationship generates a referral opportunity with probability τ.

Now, plugging in the probabilities derived previously along with the fraction of types, the market wage is [8]

$$w_M = \frac{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)} \tag{1}$$

4.2 The referral wage

For a firm making a referral wage offer, since $\alpha > \lambda_H$ there is a higher likelihood they have a link to an unemployed H type than a random selection from the pool of unemployed workers. This means that these firms will face a favourable selection from the pool of unemployed workers. Hence, this will increase the likelihood the worker hired from a referral is productive and so the expected productivity will be higher than w_M given by equation (1). A worker will reject any referral wage offer w_R less than w_M. A firm would maximize their expected profit by offering a wage $w_R = w_M$. However, due to the network formation process a firm faces a positive probability that the worker they make an offer to has more than one offer to consider. Thus this competition will bid the wages up. This bidding up process will break down at a sufficiently high wage, where a firm would have an incentive to deviate back to the lowest wage offer w_M and gamble that the recipient of their offer has no other offers to consider.

This logic will result in a symmetric equilibrium where the referral wages offered by the firms will have a probability distribution $F(\cdot)$ over the range of offers. This distribution $F(\cdot)$ is the probability that firm i's referral offer to a particular worker is higher than some other offer they have from firm j. Given the number of firms a single firm is negligible, so a firm offering a different wage cannot change the probability that an offer is accepted. This is more formally stated in the following lemma.

Lemma 1. *For any equilibrium in period 2, there must be a dispersion in the referral wages offered and there are no "gaps" in this distribution. Denote the referral wage distribution as* $F(w_R)$ *where* $w_R \in [\underline{w_R}, \overline{w_R}]$

The proof is found in appendix A. Unless otherwise stated for all further results they will also be found in the appendix. Clearly the lower bound on this distribution of referral wages offers is equal to the market wage $\underline{w_R} = w_M$, as a worker who receives less than this will reject the offer.

4.3 The expected payoff

From lemma 1 and the fact that the expected productivity of workers hired by a referral is greater than by the formal market. It must be the case that to sustain any equilibrium that all firms who make use of a referral opportunity earn the same expected profit.

$$\mathbb{E}\Pi_2(w_{Ri}) = c \;\forall\; w_R \in [w_M, \overline{w_R}]$$

Where, $\mathbb{E}\Pi_2(w_{Ri})$ denotes the expected payoff in period 2 for firm i with a referral opportunity offering wage w_R. Since, $F(\cdot)$ is the probability that firm i's offer is higher than some firm j. It must be that $1 - F(\cdot)$ is the probability that firm j offers a higher referral wage than i. Hence, in the finite case with two firms i and j with incumbent H type workers. The probability that a unemployed worker of type H with a connection to firm i receives a referral offer from firm j which is higher than i's $w_{Rj} > w_{Ri}$ is

$$\frac{\alpha}{\lambda_H 2N}\left[1 - F(w_{Ri})\right]$$

[8] In Montgomery (1991) the workers are evenly split between H and L types ($\lambda_H = 0.5$). Additionally referrals come from N firms who had an incumbent H type worker and offer a referral wage with probability τ. In other words, $\mu N = \tau N$, so $\mu = \tau$. Hence, equation (1) becomes

$$\begin{aligned} w_M^M &= \frac{\frac{1}{2}\exp\left(-\frac{\alpha\tau}{\frac{2}{2}}\right)}{\frac{1}{2}\exp\left(-\frac{\alpha\tau}{\frac{2}{2}}\right) + \frac{1}{2}\exp\left(-\frac{(1-\alpha)\tau}{\frac{2}{2}}\right)} \\ &= \frac{\frac{1}{2}\exp(-\alpha\tau)}{\frac{1}{2}\left[\exp(-\alpha\tau) + \exp(-(1-\alpha)\tau)\right]} \\ &= \frac{\exp(-\alpha\tau)}{\exp(-\alpha\tau) + \exp(-(1-\alpha)\tau)} \end{aligned}$$

Which is the same period 2 market wage derived in Montgomery (1991).

This means that the worker with an offer from firm i receives not a *single* higher referral offer from another firm is

$$\left(1 - \frac{\alpha}{\lambda_H 2N}\left[1 - F(w_{Ri})\right]\right)^{\mu N}$$

This can also be thought of as the probability a worker conditional on being an H type accepts a referral wage offer w_{Ri}. In the limit an $N \to \infty$ this becomes

$$P\left(\text{accept ref.}|H\right) = \exp\left(-\frac{\alpha}{2\lambda_H}\left[1 - F(w_{Ri})\right]\right)$$

Similar reasoning applies to the probability an L type worker accepts a referral offer. Thus, the full characterization of the equilibrium indifference condition is given by

$$\begin{aligned}\mathbb{E}\Pi_2(w_{Ri}) &= P\left(\text{Refer } H \text{ type}\right) P\left(\text{accept ref.}|H\right)(1 - w_{Ri}) + P\left(\text{Refer } L \text{ type}\right) P\left(\text{accept ref.}|L\right)(-w_{Ri}) \\ &= \alpha \exp\left(-\frac{\alpha\mu}{2\lambda_H}\left[1 - F(w_{Ri})\right]\right)(1 - w_{Ri}) - (1-\alpha)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\left[1 - F(w_{Ri})\right]\right) w_{Ri}\end{aligned} \tag{2}$$

As stated earlier this must be the same for all firms in any symmetric equilibrium for all valid wage offers w_{Ri}. Hence, a firm can deviate from its current wage offer to w_M and the expected profit should remain the same. Thus, when the referral wage is w_M, then it is guaranteed that a worker receives a higher offer from another firm. This means that $1 - F(w_M) = 1$. Hence, plugging w_M given by equation (1), into equation (2) and noting that the expected profit must equal some value c.

$$\begin{aligned}c = {} & \alpha \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right)\frac{(1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)} \\ & - (1-\alpha)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)\frac{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)}\end{aligned}$$

$$= \frac{\left[(1-\lambda_H)\alpha - \lambda_H(1-\alpha)\right]\exp\left(-\frac{\alpha\mu}{2\lambda_H}\right)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)}$$

$$= \frac{\left[\alpha - \alpha\lambda_H - \lambda_H + \alpha\lambda_H\right]}{\left[\lambda_H \exp\left(-\frac{\alpha\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)\right]\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)\exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)}$$

Which, gives the expected payoff for a firm making use of a referral referral opportunity [9]

$$c = \frac{\left[\alpha - \lambda_H\right]}{\lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)} \tag{3}$$

If a firm were to set the wage to the highest possible amount $w_{Ri} = \overline{w_R}$, then they will always secure that workers services. That is, no other firm can offer a higher wage or $1 - F(\overline{w_R}) = 0$. This combined with

[9] Substituting in the specification, where $\mu = \tau$ and $\lambda_H = 0.5$, in Montgomery (1991), equation (3) becomes the same expected payoff

$$\begin{aligned}c^M &= \frac{\left[\alpha - \frac{1}{2}\right]}{\frac{1}{2}\exp\left(\frac{(1-\alpha)\tau}{2(1-\frac{1}{2})}\right) + (1-\frac{1}{2})\exp\left(\frac{\alpha\tau}{2\frac{1}{2}}\right)} \\ &= \frac{\frac{1}{2}\left[2\alpha - 1\right]}{\frac{1}{2}\exp\left((1-\alpha)\tau\right) + \frac{1}{2}\exp\left(\alpha\tau\right)} \\ &= \frac{\left[2\alpha - 1\right]}{\exp\left((1-\alpha)\tau\right) + \exp\left(\alpha\tau\right)}\end{aligned}$$

the fact that the expected payoff is the same for all referral wage offers means that equation (2) becomes

$$\begin{aligned} c &= \alpha(1-\overline{w_R}) - (1-\alpha)\overline{w_R} \\ &\Leftrightarrow \\ \overline{w_R} &= \alpha - c \end{aligned} \tag{4}$$

That is the worker offered $\overline{w_R}$ is of H type with probability α and the expected payoff is $\alpha - \overline{w_R}$.

4.4 Value of a referral opportunity

The model so far has generalized that of Montgomery (1991) by allowing an arbitrary mass of firms who can make use of a referral μ and the proportion of H and L type workers to vary rather than be fixed. It is still the case that the possession of a referral opportunity is valuable, moreover, the value c is determined by equation (3). On the formal market the firms compete away any expected profits. Hence, a positive expected profit is earned only by a firm who can make use of a referral and so is able to increase the likelihood that they will hire a productive worker.

Where my model differs is in the behaviour of the value of a referral opportunity c. In Montgomery (1991) this value is given by

$$c^M = \frac{2\alpha - 1}{\exp\left((1-\alpha)\tau\right) + \exp\left(\alpha\tau\right)}$$

It can be shown that c^M is always increasing in α. Whereas, for my more general specification for the value of a referral opportunity this is not necessarily the case. In fact, when the proportion of H type workers λ_H is sufficiently small and the mass of firms with a referral opportunity μ is large enough, then the value will decrease as the signal α increases. This can be thought of the signal quality improving. For example if $\alpha = 1$ then a firm knows that they have found a H type for sure. This is because when there is a scarcity of H type workers and there are many firms who have the opportunity to make use of a referral μ is large. Even as the signal quality increases the competition is fierce. An H type worker is likely to receive numerous offers. As there is no limit on the number of connections they can have whereas, firms are restricted to one. Hence, for a single firm it is unlikely that they will have their offer accepted unless they offer higher wages to attract the worker. So they extract a lower expected profit. Hence, the value the referral opportunity c is diminished. Yet, with the same proportion of worker types but relatively fewer firms with the opportunity to make use of a referral i.e. μ is small. Then, an increase in α is valuable to such a firm, as there is not much competition and so it is very likely that their referral wage offer is received by an H worker who will accept it.

To summarise if there is lots of competition i.e. μ is high and relatively few H types then improving the signal quality will actually make the firms worse of in terms of their expected profit. In contrast, if there is not much competition i.e. μ is low then improving the signal quality benefits the firm. One question that might be raised by this is what determines the number of firms μ who can make a referral offer? This is the focus of the next section.

4.5 Number of referral opportunities

Montgomery (1991) simply exogenously allocates workers to firms before they hire workers. This prevents any possible answer to question of what determines the mass of firms μ with a referral opportunity. I will introduce a simple demand and supply analysis to endogenously determine this mass of firms μ. In this model the "quantity" dimension of a referral opportunity is given by the parameter μ[10]. Whilst, the "price" dimension corresponds to the value of the referral opportunity given by equation (3).

$$\text{Price of referral} = c(\mu) = \frac{\alpha - \lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)} \tag{5}$$

The function $c(\mu)$ is clearly a decreasing function of μ and so the above equation describes a downward sloping demand curve for referral opportunities.

Now, in period 1 firms can pay a cost κ to enter and access a single referral opportunity (to potentially be used in period 2) by hiring a single worker. This cost can be thought of as an investment in a recruitment technology that their incumbent can use in period 2. If firms do not enter in period 1 then

[10] Note that earlier I used N to scale the overall population size of firms and that μN was the total number of single vacancy firms with a referral opportunity.

they can still hire a single worker in period 2 via the formal method, where the expected profit is equal to zero given the zero enter condition $\mathbb{E}\Pi_1(\text{no entry}) = 0$. If they do pay the cost to enter in period 1 then as per the details in section 3.3 their expected profit is given by

$$\mathbb{E}\Pi_1(\text{entry}) = \lambda_H(1 + c(\mu) - w_{M1}) + (1 - \lambda_H)(c(\mu) - w_{M1}) - \kappa$$

Since firms are assumed to pay the expected productivity $w_{M1} = \lambda_H$, the above equation reduces to $\mathbb{E}\Pi_1(\text{enter}) = c(\mu) - \kappa$. In any equilibrium a firm must be indifferent between entry and no entry. Equating the expected profits gives $c(\mu) = \kappa$. The equilibrium mass of referral opportunities is given by the μ satisfying $c(\mu) = \kappa$. The mass of firms μ can be thought of as the mass of firms with a referral opportunity. The following proposition fully describes the equilibrium.

Proposition 1. *The equilibrium entry of firms μ^* is derived from solving the condition $c(\mu) = \kappa$ and is characterized by the two following cases. Let the cost of a referral be positive $\kappa > 0$, then:*

1. *For a sufficiently small cost a mass of firms μ^* use referrals to hire workers. The workers who are not hired by a referral are hired in the formal market.*
   - *That is if $0 < \kappa < \alpha - \lambda_H \;\Rightarrow \mu^*(\alpha, \lambda_H, \kappa) \in (0, \infty)$.*
2. *For a sufficiently large cost no firms use referrals, they all hire in the formal market.*
   - *That is if $\kappa \geq \alpha - \lambda_H \;\Rightarrow \mu^*(\alpha, \lambda_H, \kappa) = 0$.*

If the cost is sufficiently high then intuitively the benefit of using a referral e.g. access to a more favourable pool of workers, is not outweighed by the cost, that is they incur negative expected profits. Naturally firms would rather earn zero than negative expected profits. So they hire in the formal market, with zero expected profits due to free entry.

For a sufficiently small cost, there is a μ that satisfies $c(\mu) = \kappa$. This means that accessing the referral opportunity will generate them positive expected profits, equal to exactly κ. Crucially the condition $\alpha > \lambda_H$ means that for these firms the probability of finding an unemployed H type via a referral is greater than in the formal market [11]. However, for the other firms currently hiring in the formal market, if they entered to access a referral. The mass of referral opportunities would increase but the value of a referral would decrease to less than the cost. Hence, it would not be a profitable strategy for them. Instead they don't pay the cost to enter and they accept hiring via the formal market.

The entry cost k is in essence a horizontal supply curve for employment and so referral opportunities. This is the most simple specification. It could of course be easily accommodated to specify it as an upward sloping supply curve.

Finally, the model can be adapted to incorporate employment opportunities q[12] rather than the mass of referral opportunities μ. Suppose that each employment relationship generates a referral opportunity with probability τ. This generates the equilibrium condition $c(\tau \times q) = \kappa$.

4.6 Comparative statics

In this section I will document how the endogenous variable of interest μ^* changes in response to changes in the primitives of the model α, λ_H and κ. Define the following two equations

$$\phi^*(\alpha, \lambda_H, \kappa) \equiv (1 - \lambda_H) \exp\left(\frac{\alpha\mu^*(\alpha, \lambda, \kappa)}{2\lambda_H}\right) \frac{1}{2\lambda_H} \tag{6}$$

$$\gamma^*(\alpha, \lambda_H, \kappa) \equiv \lambda_H \exp\left(\frac{(1-\alpha)\mu^*(\alpha, \lambda, \kappa)}{2(1 - \lambda_H)}\right) \frac{1}{2(1 - \lambda_H)} \tag{7}$$

The following lemma summarises how the mass of referring firms μ^* responds to changes in the primitives of the model.

[11] If no firms used a referral then they are all hiring in the formal market where they uniformly at random select a single worker and hire them. The probability they select an H type is simply λ_H.

[12] As in Montgomery (1991).

Lemma 2. *For an appropriate cost $\kappa \in (0, \alpha - \lambda_H)$. The, response of μ^* with respect to changes in the parameters $\{\alpha, \kappa, \lambda_H\}$ are*

$$\frac{d\mu^*}{d\alpha} = \frac{1 - \mu^* \kappa \left(\phi^* - \gamma^*\right)}{\kappa \left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} \tag{8}$$

$$\frac{d\mu^*}{d\kappa} = \frac{-2\left(\lambda_H \phi^* + (1-\lambda_H)\gamma^*\right)}{\kappa \left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} \tag{9}$$

$$\frac{d\mu^*}{d\lambda_H} = \frac{\left[\phi^*\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma^*\left(2(1-\lambda_H)^2 + \lambda_H(1-\alpha)\mu^*\right)\right]}{\lambda_H(1-\lambda_H)\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} - \frac{1}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} \tag{10}$$

The following proposition summarises the conditions under which entry is increasing or decreasing.

Proposition 2. *For an appropriate cost $\kappa \in (0, \alpha - \lambda_H)$. The conditions under which the mass of referring firms μ^* is increasing or decreasing are:*

- *The signal quality*
  - $\frac{d\mu^*}{d\alpha} \geq 0 \Leftrightarrow 1 \geq \mu^* \kappa \left(\phi^* - \gamma^*\right)$.
  - $\frac{d\mu^*}{d\alpha} < 0 \Leftrightarrow 1 < \mu^* \kappa \left(\phi^* - \gamma^*\right)$.
- *Cost of entry*
  - $\frac{d\mu^*}{d\kappa} < 0 \; \forall \alpha, \kappa, \lambda_H$
- *Quality of candidates*
  - $\frac{d\mu^*}{d\lambda_H} \geq 0 \Leftrightarrow 1 \leq \frac{\kappa\Psi^*}{\lambda_H(1-\lambda_H)}$.
  - $\frac{d\mu^*}{d\lambda_H} < 0 \Leftrightarrow 1 > \frac{\kappa\Psi^*}{\lambda_H(1-\lambda_H)}$.

Where, $\Psi^* = \left[\phi\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma\left(2(1-\lambda_H)^2 + \lambda_H(1-\alpha)\mu^*\right)\right]$

Proof. From the results in lemma 2 simply check the conditions for which, if indeed it is at all, the derivative of interest is positive, negative or zero. For a particular parameterization of the model the exact conditions for when the derivative of μ with respect to α or λ_H is increasing or decreasing is quite cumbersome, but can be derived from a numerical simulation of the model. □

This proposition shows that the mass of referring firms μ^* is always decreasing as the cost entry increases, as would be expected. However, with respect to the signal α and the proportion of H types λ_H, there is a non-monotonic relationship. In both cases entry is increasing up to a point after which it decreases. Taking λ_H for example, although it is not obvious from the equation, for a given α and κ when there are relatively few H type workers then the mass of referring firms μ^* is increasing in response to an increase in λ_H. The increase in the proportion of H types increases the likelihood of the referral going to an unemployed H worker. This induces the entry of firms into accessing referral opportunities. Yet, as λ_H continues to increase the number who do not receive a referral offer, due to the stochastic nature of the social link formation, increases. Hence, the expected productivity of workers in the formal market increases driving up the formal market wage. As competition is high firms would have to offer unprofitably high wages to attract workers and so fewer are willing to pay the cost and μ^* decreases.

4.7 Hiring probabilities

In the empirical literature on referral hiring there is a lot of heterogeneity in the fraction being hired via this method. Ioannides and Datcher Loury (2004) use the 1994 USA Current Population Survey (CPS) to determine how people obtained their current job. They show that for the unemployed, those with fewer years of education, found jobs through a referral at a relatively higher rate to those with more years of education. However, for currently employed workers seeking a new job this is reversed. Elliott (1999) shows that the fraction finding a job through a referral increases with the size of the city. Galenianos (2014) documents that the fraction varies by industry in the USA. So far the studies have focused on the USA. However, heterogeneity in the fraction being hired from a referral are also documented in other parts of the world. For instance, Pellizzari (2010) also shows that in Europe the fraction is greater for workers with fewer years of education. He also documents that the fraction varies by country, with higher fractions in southern European countries such as Portugal and Spain as compared to northern ones such as the UK and the Netherlands.

All of these studies indicate that the probability of an individual worker being hired by a referral can vary. The aim of this paper is not to explain all the possible variations in a single model. Instead, it is to note that as far as I know the literature has not fully explored the possible determinants of this heterogeneity. Models which incorporate referrals in employee hiring, often exogenously allocate the initial workers to the firms [13]. However, this seems to be a strong assumption as it does not say anything about how the firm came to hire their initial workers. Moreover, assuming that referrals are valuable it becomes clear that understanding how the competition for them impacts their frequency of use is of first order interest.

Another aspect that is important to understand is the cost of a referral. In the Type-Independent case the cost can be thought of as the investment by the firm in a technology to asses the ability of a new employee. How does this cost interact with the prevalence of referrals both directly and indirectly? In summary, by allowing the amount of referral opportunities to be endogenously determined. I hope to shed some light on the conditions under which their prevalence is high or low.

As there is no unemployment in this environment, for any worker there are only two possible hiring outcomes. They are either hired by a referral, denote that by R. Or they are hired in the formal market e.g. not fired by a referral, denote that by M. Section 4.1 details how to determine the probability of being hired in the formal market conditional on both unemployed worker types. Given the underlying distribution of the types of workers the probability of being hired in the market is given by

$$\begin{aligned} P_M &= P(M|H)P(H) + P(M|L)P(L) \\ &= P(M|H)P(H) + P(M|L)(1 - P(H)) \\ &= \exp\left(-\frac{\alpha\mu^*(\alpha,\lambda,\kappa)}{2\lambda_H}\right)\lambda_H + \exp\left(-\frac{(1-\alpha)\mu^*(\alpha,\lambda,\kappa)}{2(1-\lambda_H)}\right)(1-\lambda_H) \end{aligned}$$

This is for both types of unemployed workers. As a consequence the probability of being hired by a referral is given by $P_R = 1 - P_M$, which gives

$$P_R = 1 - \left[\lambda_H \exp\left(-\frac{\alpha\mu^*(\alpha,\lambda,\kappa)}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*(\alpha,\lambda,\kappa)}{2(1-\lambda_H)}\right)\right] \tag{11}$$

The following section discusses numerical results about the probability of being hired by a referral. The numerical exercise suggests that there is an interesting non-monotonic relationship in the probability of being hired by a referral and the signal quality. In particular when the cost of a referral κ is low then increases in the signal quality α can lead to the probability decreasing. However, for higher costs the relationship is the inverse and the probability increases. The intuition if the following. For low κ competition in referrals is high not because the signal quality α is high but because the barriers to entry are low. As α increases fewer firms are willing to pay the cost as they know that there are a fixed number of H types. The higher the signal quality the lower the noise around that true number of H types. However, for a higher κ competition is low and so any improvements in the signal are beneficial to the firm and so they are willing to pay the cost and this increases the probability. The following lemma characterizes the non-monotonicity of the probability of being hired by a referral in this economy. Using the definitions of $\phi*$ and γ^* given by equations (6) and (7) respectively.

Proposition 3. *Fix λ_H:*

1. *For a sufficiently low cost $\kappa^-(\kappa \to \gamma,\ \gamma > 0)$ there exists a point $\bar{\alpha}$ such that probability of being hired by a referral is increasing in the signal quality for $\alpha < \bar{\alpha}$ and decreasing for $\alpha > \bar{\alpha}$.*

2. *For a sufficiently high cost $\kappa^+(\kappa \to \alpha - \lambda_H)$ the probability of being hired by a referral is always increasing in the signal quality α.*

The intuition for this is the following. When the cost is high enough κ^+, then the initial level of competition for referrals is low. This results in fewer workers being hired by a referral. In particular the fraction of workers being hired is less than the underlying fraction of productive H types workers. Since referrals give an informational advantage this is lower than it would be if there was no cost associated to using a referral. Thus any increase in the signal α, will increase their certainty of being referred a

[13] See for example Montgomery (1991) and Casella and Hanaki (2008).

productive worker and so will induce more firms to access the referral market. This raises the competition for referrals but since the H types are being hired by referrals at a lower rate than they should be, the fraction of workers being hired will increases even with this increase in competition.

Conversely, when the cost is low enough κ^- and there is sufficient informational advantage to using referrals. Then the initial level of competition for referrals is high. Many firms are using referrals and as a consequence the fraction of workers being hired is less than the underlying fraction of productive H types workers. Although, there is some informational advantage to hiring by a referral, there is enough uncertainty in the signal to sustain this level of competition. An increase in α reduces this uncertainty of the type of worker they will be referred and so induce fewer firms to access the referral market. This decreases the competition for referrals and so the fraction being hired by a referral will decrease.

Figure 1 shows a non-monotonicity in the relationship between the fraction of workers hired by referrals $P(R)$ and the signal quality α. As can be seen by difference the shape of the curves in the left and right hand panels. The role of the cost of referrals is of interest. For instance, on the left hand panel the cost κ is low and as the signal quality increases the probability of being hired by a referral increases rapidly above the underlying fraction of H types (indicated by the dashed red line) and then decreases to below that level. Whereas, for the right hand panel as the signal quality increases the probability is increasing and always below the fraction of H types.

If $\alpha - \lambda_H \leq \kappa$ then only the formal hiring method is used. However, as the signal quality becomes better $\alpha - \lambda_H > \kappa$ then it is worthwhile for firms to acquire the technology to access the informal hiring method. There will be a rapid increase in the fraction of workers who are hired by referrals. The signal quality is low $\alpha - \lambda_H \simeq \kappa$ creating uncertainty as to the identity of the recipient of the referral. This uncertainty at sufficiently low costs sustains higher levels of firms, hence μ^* is higher and the fraction of workers hired by a referral is higher.

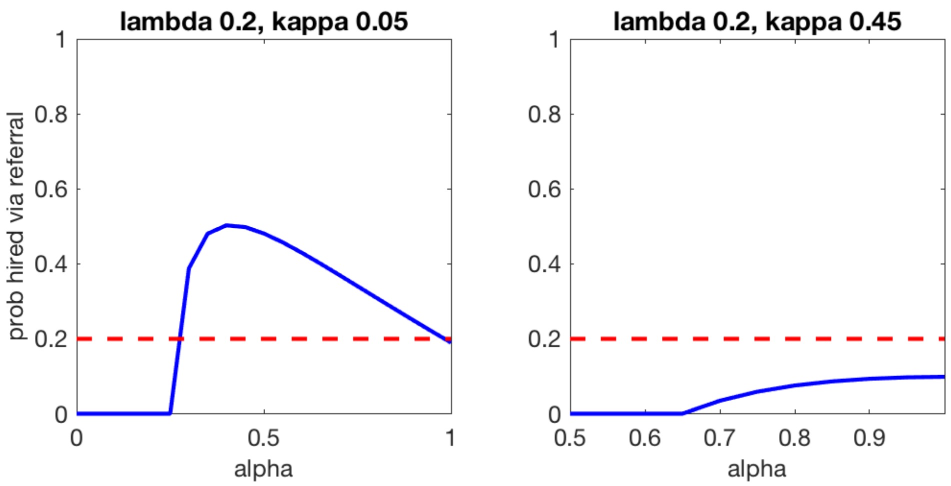

Figure 1: The probability of being hired via a referral as function of α

In order to better understand this dynamic imagine if the signal quality was perfect $\alpha = 1$. This means a firm with the technology knows for sure if they have a connection to an H type. If κ is small it is worthwhile for many firms to pay the cost to acquire the technology and try to hire an H type. Then given that the underlying distribution of H types is λ_H in equilibrium, the mass of firms with a referral opportunity μ^* would be such that the probability of a worker being hired via a referral is close[14] to this underlying distribution. Imagine under this scenario if the probability of being hired by a referral was higher than λ_H. Then since firms have a perfect signal they know there would be a positive probability of hiring unproductive L types and they would have an incentive to deviate and not enter in period 1. Also when the cost κ is low the probability of being hired by a referral is high but competition for referrals is also high, unemployed workers will be receiving many more offers. As α increases after a point the fraction hired by referral decreases the competition decreases. This may seem counter-intuitive as increasing the signal quality α gives the firm more information about their potential new employee. But by increasing the signal quality the uncertainty of the employees type is reduced. This brings the level

[14]It will not necessarily be exactly the same due to the positive cost κ. The smaller the cost the closer that the probability of being hired by a referral is to the underlying distributions of H types.

of competition back to the "correct" level given the underlying distributions of types.

For higher costs κ the opposite occurs. Now increases in α increase the probability a worker is hired by a referral. The reason is that due to the high barriers to entry from the relatively high cost κ competition is low and any improvements in α make it worthwhile for a firm to acquire the technology. The fraction of workers being hired by a referral is below the underlying distribution and increases in the signal quality will reduce the uncertainty in the informal hiring method. As before due to the presence of a positive cost the actual fraction of workers hired by a referral will be slightly below the true distribution of H type workers.

Figure 2 also shows this phenomenon from a different perspective. It plots for given α and λ_H as κ increases. The likelihood of a worker being hired via a referral decreases as κ increases. This is in line with the economic principle of demand, the higher the cost for referrals the lower the demand and hence the less likely it is for a worker to be hired by a referral. The right hand panel has a higher α than the left hand panel. The impact of this is to shift the curve to the right and rotate it counter clockwise.

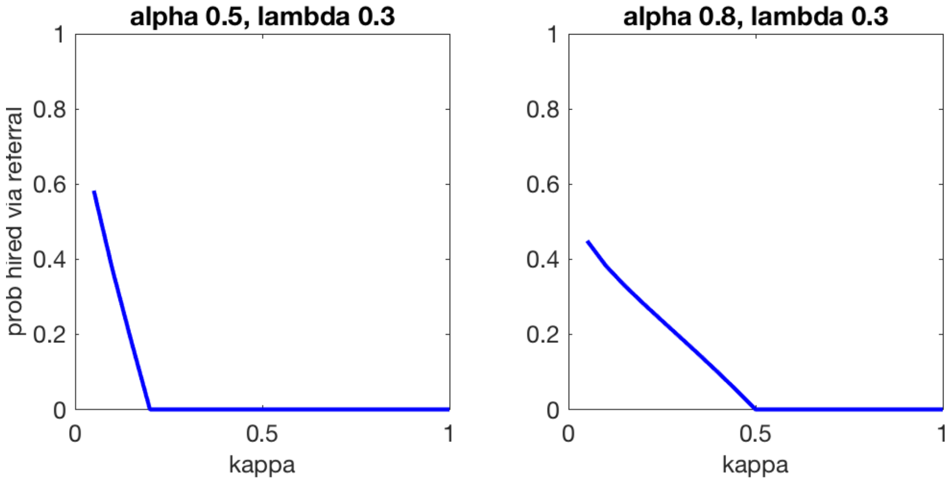

Figure 2: The probability of being hired via a referral as function of κ

Figure 3 shows that the probability of being hired via a referral is non-monotonic in λ_H. In a market with relatively few quality candidates, the likelihood of a firm hiring an H type via a referral is low. The unconditional probability of connecting to a H type is α and so remains fixed. Yet the relatively scarcity of H types means that those workers will have many links and so many offers to choose from. For any individual firm the probability of them hiring an H type is low. Firms optimally respond to this by not entering the market for referrals and instead hiring in the formal market. As λ_H increases, the probability that firms can hire an H type increases as there are more workers to potentially connect to. Thus the likelihood a worker is hired via a referral increases. Of course this then attracts more firms increasing the competition for these workers and so reducing the value of a referral to a firm, Eventually, the likelihood of a worker being hired via referral decreases steeply as the benefits of a referral are not outweighed by the costs.

The signal α is important. The left hand panel is for a lower α than the right hand panel and it is clear that the values of λ_H that sustain referrals is greater. Also the highest probability of being hired via a referral is greater for a higher α. This makes sense as the more likely a firm with a referral opportunity is to connect with an H type the more valuable the referral is to them.

4.8 The distribution of referral wages

This distribution $F(\cdot)$ is the probability that firm i's referral offer to some woker is higher than some other firm j. Taking the equation from equation (2), equating it with the cost of entry κ and assuming that $0 < \kappa < \alpha - \lambda_H$ gives the following indifference equilibrium condition, where the index i is dropped

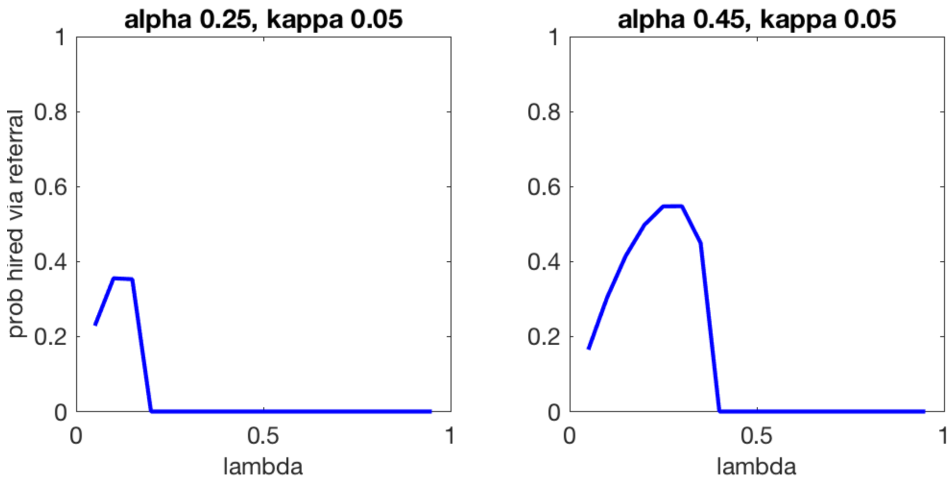

Figure 3: The probability of being hired via a referral as function of λ_H

as the strategy of the firms is symmetric

$$\alpha \exp\left(-\frac{\alpha\mu^*(\alpha,\lambda_H,\kappa)}{2\lambda_H}\left[1-F(w_R)\right]\right)(1-w_R) \\ -(1-\alpha)\exp\left(-\frac{(1-\alpha)\mu^*(\alpha,\lambda_H,\kappa)}{2(1-\lambda_H)}\left[1-F(w_R)\right]\right)w_R=\kappa \quad (12)$$

Inspecting the above equation it is clear that there are two unknowns as they are not a function of the parameters, the distribution F and the possible referral wage w_R. However the equation can be simplified with the upper and lower bounds of the support of the distribution F. Namely, the lowest (also referred to as the market wage) and highest referral wage given by given equations (1) and (4) respectively. In both cases the distribution F is the only unknown variable and so I have a single equation with one unknown variable. So for each valid parametrization there is a unique F that solves equation (12). Unfortunately it is not possible to find a closed form solution of F. Instead I can approximate the distribution by numerically solving for F.

To do this I will proceed with the following steps. Fix the parameters α, λ_H and κ and solve $c(\mu)=\kappa$ given by equation (5) for the unique $\mu^*(\alpha,\lambda_H,\kappa)$. Find w_M and $\overline{w_R}$ for the same parameterization. Pick a $w_R^* \in (w_M, \overline{w_R})$ near the lower bound w_M and substitute this into equation (12) to solve for $F(w_R^*)$. Repeat this process, increasing w_R^* incrementally each time until reaching the upper bound $\overline{w_R}$.

This will generate an approximation for the distribution F. This will allow me to explore questions such as how do changes in the composition of workers affect the expected wage or the probability that a firms referral offer made to a worker is the highest that the worker will receive.

Definition 1 (First order stochastic dominance)**.** Distribution H first order stochastically dominates distribution $G \iff H(x) \leq G(x)\ \forall x \in X$.

Proposition 4. *Let $F(\cdot)$ and $F'(\cdot)$ be two distributions for parametrization $\{\alpha,\lambda_H,\kappa\}$ and $\{\alpha,\lambda'_H,\kappa\}$ respectively. Then by definition 1, Distribution F' first order stochastically dominates distribution F, or equivalently*

$$F'(w_R) \leq F(w_R)\ \forall w_R$$

This proposition says that as the proportion of H types increases, whilst the other factors (signal and cost of entry) remain fixed. For a firm offering a referral wage offer w_R to a worker k. The probability that this offer is the highest that worker k receives is lower compared to a the same offer in a market with fewer H types. In other words if a firm wants to have the same probability that their referral offer is the best their incumbent employee passes on to their unemployed contact.Then in a market with relatively more H type workers they must increase their referral wage offer. This effect can be seen in figure 4.

Improving the productivity (by increasing λ_H) of the workforce increases the expected wages for all workers. The highest referral wage remains unchanged but the market wage increases. This can be seen easily since the highest referral wage is given by $\overline{w_R}=\alpha-c$ and in equilibrium $c=\kappa$. Hence, there is no

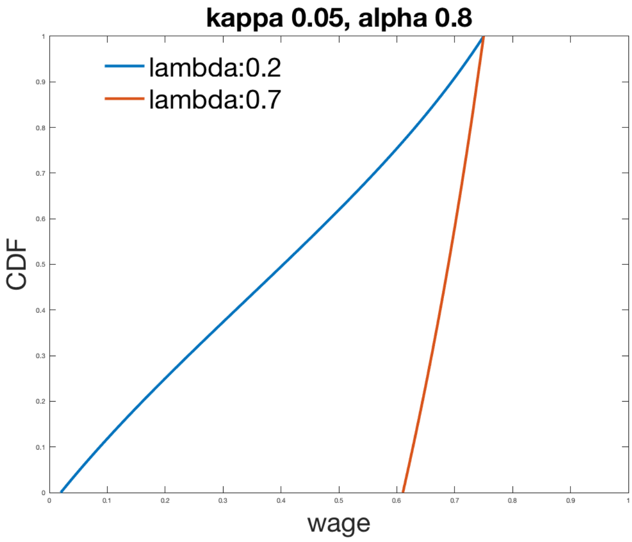

Figure 4: The distribution F as λ_H increases.

effect on the highest referral wage from an increase in λ_H. For the formal market wage the calculations are more complicated to show the direction of the effect. However, the intuition for the decrease is the following. As λ_H increases there are now relatively more productive workers in the workforce. From proposition 2 if λ_H is sufficiently low firms will respond to this by entering the market for referrals. Since the probability a firm makes a referral offer to an productive H type is held constant at α. This increased competition will drive up the equilibrium wage offers in order to attract the H type workers. Proposition 2 also tells us that if λ_H is sufficiently high then firms will respond by exiting the market for referrals. However, now that the pool of workers is largely made up of H types, the expected productivity of the workers not hired by a referral will increase, as not all can be hired by a referral if few firms are using them. Hence, the formal market wage increases as workers have a higher outside option.

Whilst this is the case for increasing the productivity of the pool of workers λ_H. This is not the case with increases in the signal or the cost of entry. Instead increasing either of these parameters, whilst holding the others constant, will lead to a rotating of the distribution F.

Proposition 5. *For any two distributions parameterized by different values for either α or κ, whilst holding the other parameters constant. Then first order stochastic dominance of one distribution over the other can not be achieved.*

Inspecting figure 5, it is clear that an increase in α rotates the distribution F in a clockwise fashion. Whereas, an increase in κ leads to a anticlockwise rotation. As stated earlier in equilibrium the highest referral wage is given by $\overline{w_R} = \alpha - k$. So clearly, an increase in the signal quality α increases $\overline{w_R}$. This is because the firms are more likely to hire an H type. Whereas, an increase in κ decreases $\overline{w_R}$. Now, less firms are willing to use a referral and so the competition for the workers decreases, driving the highest wage down. For the formal market wage as α increases, but importantly the distribution of types remains fixed, then a firm is more likely to connect to an H type. This means that relatively more L workers will not have a link to a firm. This will drive down the expected type in the formal market which reduces the expected productivity of that market[15]. Hence, the formal market wage decreases. In contrast, when κ increases. Fewer firms will use a referral, so there will be fewer social links between H types and firms. Hence, the expected productivity of the formal market increases and so does the wage.

A potential economic interpretation of propositions 4 and 5 is to do with wage inequality. Let a measure of wage inequality be the difference between the highest referral wage and the formal market wage. Where the larger the difference the greater the level of inequality. The results of this section suggest that increases in signal quality α will increase inequality. Whereas, an increase in the proportion of H types or decreases in the costs of a referral κ will reduce inequality. Note that conversely inequality, by this measure, can be reduced by reducing α. However, as the left hand panel of figure 5 clearly demonstrate this occurs by reducing the highest referral wage as well as increasing the formal market

[15]This process is a form of the lemons selection effect.

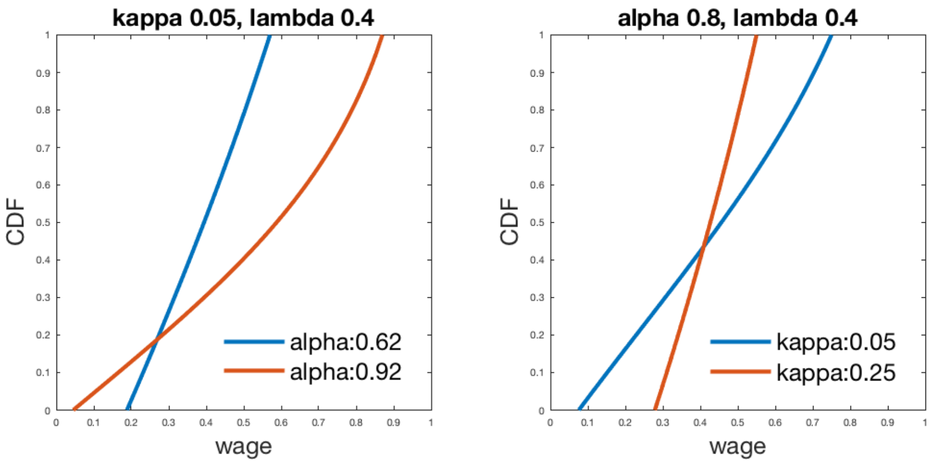

Figure 5: The distribution F as α and κ increase.

wage. Hence, it reduces inequality by making some workers better off but some worse off. Whereas, with respect to λ_H the improvement is unambiguously better for all participants.

5 Type-Dependent Equilibrium

In order to determine the equilibrium for this specification the steps are the same as section 4. The main difference will be how the equilibrium is stated and the steps need to prove the equilibrium is unique.

In this section if a firm pays a cost κ to enter the period 1 market to hire a worker in order to access the informal hiring method. Then the signal they receive in period 2 about a potential unemployed worker is dependent of the type of worker they had hired. If they hired an H type in period 1 then in period 2 the signal $\hat{\alpha}$ tells them the probability they have link through their incumbent employee to another H type.

The following proposition outlines the equilibrium for the case where these is some dependence between types.

Proposition 6. *The equilibrium entry of firms μ^* is derived from solving the condition $c(\mu) = \kappa$ and is characterized by the two following cases. Let the cost of a referral be positive $\kappa > 0$, then:*

1. *For a sufficiently small cost a mass of firms μ^* enter period 1 for access to a referral in period 2. However, if there is homophily in the social network then only those firms who hired an H type worker make use of a referral. The workers who are not hired by a referral are hired in the formal market.*
   - *That is if $0 < \kappa < \hat{\alpha} - \lambda_H$ and $1 - \hat{\alpha} < \lambda_H \Rightarrow \mu^*(\hat{\alpha}, \lambda_H, \kappa) \in (0, \infty)$.*
2. *For a sufficiently large cost no firms use referrals, they all hire in the formal market.*
   - *That is if $\kappa \geq \hat{\alpha} - \lambda_H \Rightarrow \mu^*(\hat{\alpha}, \lambda_H, \kappa) = 0$.*

Here in the informal hiring method only firms who have hired a productive worker in period 1 will make use of a referral in period 2. There are analogous results as in section 4. The main difference is that the type-dependency imposes greater restriction on the use of referrals. For example comparing figure 6 to figure 1. It is clear that probability is decreasing everywhere for a sufficiently low cost but increasing everywhere for a sufficiently high cost.

6 Conclusion

I build on the framework of Montgomery (1991) by introducing some element of competition between firms for access to the valuable "option" of hiring by a referral. I study a general case of a recruitment technology that allows a firm to access better information about a new potential employee relative to a competitive market. This technology cost is captured by κ. In a labour market there are a large

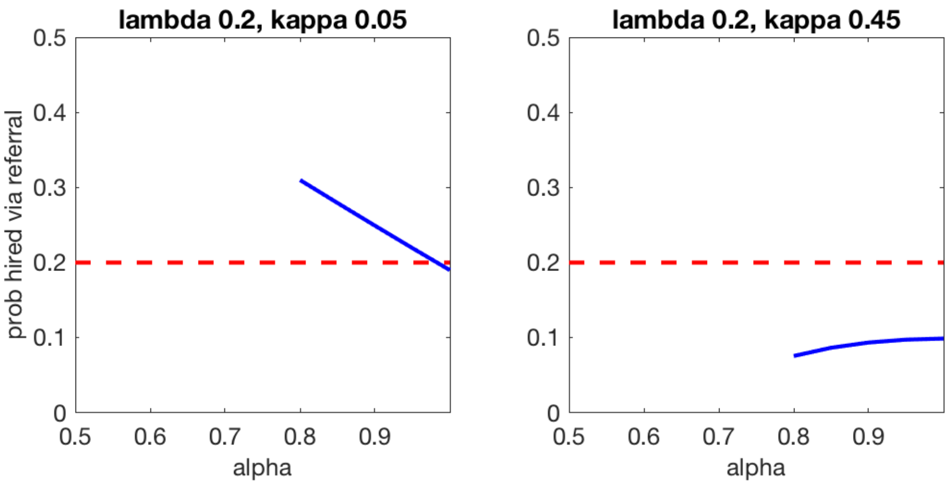

Figure 6: The probability of being hired via a referral as function of $\hat{\alpha}$ with type-dependency

number of unemployed workers of two types of workers; productive H types and unproductive L types. A fraction λ_H are H types and $1-\lambda_H$ are L types. There are two periods. In the first period firms hire a worker and in period 2 there are two overlapping generations of workers, the incumbents hired in period 1 who are retiring and a new generation born this period. In period 2 the firm must replace their retiring worker and there are two methods to do this. Either via an formal method or an informal method. The formal method is the competitive market and the informal is a referral used in conjunction with the recruitment technology. I look at two cases of this recruitment technology in particular. In the first case the recruitment technology efficacy is independent of the type of the incumbent worker. Whereas, in the second case there is some correlation between the ability of the incumbent to use the recruitment technology to identify a productive unemployed worker and their type. Both cases create a favourable selection from the pool of unemployed workers as they are more likely to hire a productive H type via a referral. Hence, firms earn a positive expected profit c from this strategy, given by equation (3). For those firms who do not have the opportunity to use a referral, they can hire in the formal market. This happens after workers have been hired by a referral. Hence, there is a "lemons" effect, as this pool of workers are on average less productive than the pool or referred workers. There is a free entry condition in the formal market and so expected profits are zero.

Through a simply demand-supply analysis I determine the equilibrium quantity μ^* of referrals. Equation (5) represents the demand of for referrals, denoted $c(\mu)$ and κ is a horizontal supply curve. Hence, the equilibrium μ is determined for a particular parameterization where demand equals supply or $c(\mu^*) = \kappa$.

I then conduct some comparative statics on this equilibrium object and show how it changes in response to changes in the parameters α, κ and λ_H. I show that the amount of referrals is decreasing in the cost but non-monotonic in the proportion of H types and the signal. I also construct the probability of being hired given by equation (11). Since there is no closed form solution for μ^* I numerically estimate it. I then plot how this probability evolves with changes in the underlying parameters of the model. The most interesting finding is that as α increases, the likelihood of being hired by a referral can be increasing or decreasing depending on the cost κ. When κ is sufficiently low then entry is quite high and increasing α makes it more likely that a firm connects to an H type. However, given the high level of entry and that workers are not restricted to a single link then firms would need to compete via wages to attract workers. Hence, the marginal benefit is greater than the marginal cost, so firms exit. This reduces the likelihood of being hired via a referral. On the other hand for sufficiently high κ the reverse is true since there a relatively few firms to compete with and so the likelihood increases.

Finally I compute the distribution F numerically, which is the probability that firm i's referral offer to some worker is higher than some other firm j. I then show that, holding all else constant, only increases in λ_H lead to first order stochastic dominance. Defining wage inequality as the difference between the highest referral wage and the formal market wage. Then, whilst increases in κ or decreases in α reduce inequality, they achieve this by simultaneously reducing the highest referral wage and increasing the formal market wage. Whereas, due to the first order stochastic dominance, increases in λ_H reduce inequality by only increasing the formal market wage. This potentially has implications for policy on wage inequality in markets where referrals are used.

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A Proofs

This section details the proofs for the propositions of the paper

Lemma 1

Proof. To establish wage dispersion I will make use of a result in Burdett and Judd (1983). From the details of the network formation in section 3.4, firms with a connection do not know the exact number of referral wage offers the young worker they are connected to will receive. However, they do know the probability of any particular number of wage offers a worker can receive. Thus, the firms are bidding for the workers labour, yet they don't know how many other firms are also bidding for the workers labour. Workers are effectively searching for the best wage amongst their offers. This type of search is referred to as noisy search. Noisy search coupled with the fact that the probability that a worker receives 1 offer ($\alpha \times \frac{1}{\lambda_H 2N}$) is in the interval $(0, 1)$ means that, according to Theorem 4 of (Burdett and Judd (1983)) the unique equilibrium of wage offers must be dispersed.

Furthermore, there is a lower and upper referral wage offer. The lower referral wage offer exists because workers can get hired in the market if they refuse their wage offers, so they have an outside option. The upper referral wage offer exists because any wage offer above it will not increase the likelihood of a worker accepting it, hence it does not increase the expected probability of hiring a worker but only increases the firms costs, through wages. So expected profits would decrease.

This means that for any equilibrium in period 2 the referral wage offers have some distribution $F(w_R)$ where $w_R \in [\underline{w_R}, \overline{w_R}]$.

Additionally, to show that there are no gaps in this distribution I will use proposition 2.2 of Butters (1977). Suppose, that there were two referral wages offered w_1 & w_2 to a single worker by firm 1 and firm 2 respectively. Furthermore, let $\underline{w_R} < w_1 < w_2 < \overline{w_R}$. The firm 2 could reduce its wage offer by $\epsilon > 0$ such that it is still higher than firm 1, that is $w_2 - \epsilon > w_1$. This means that firm 2 can ensure they still hire the worker without affecting the probability of their wage being accepted and reduce costs. Increasing their expected profits. Hence there can be no gaps. □

Proposition 1

Proof. I will show that for a positive cost of entry κ that is sufficiently small there is a unique μ that solves for following equation

$$\frac{\alpha - \lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)} = \kappa \tag{13}$$

This means that there is a positive mass of referral opportunities. However, for large enough κ there is no solution and hence no firms use the referral opportunities. Denote the left hand side $c(\mu)$ and on one hand $\mu = 0$ gives

$$\begin{aligned} c(0) &= \frac{\alpha - \lambda_H}{\lambda_H \exp(0) + (1-\lambda_H)\exp(0)} \\ &= \frac{\alpha - \lambda_H}{\lambda_H + (1-\lambda_H)} \\ &= \alpha - \lambda_H \end{aligned}$$

On the other hand as $\mu \to \infty$, $c(\mu)$ becomes

$$\lim_{\mu\to\infty} c(\mu) = 0$$

Hence, $c(\mu)$ is a downward sloping curve as a function of μ and it's maximum value is $\alpha - \lambda_H$. The right hand side of equation (13) is κ, which is constant. Therefore, there is a $\kappa \in (0, \alpha - \lambda_H)$ such that μ^* solves equation (13). For $\kappa > \alpha - \lambda_H$ there is no point at which the left and right hand side of equation (13) are equal, so no firms will access the referral market $\mu^* = 0$. Also, at $\kappa = \alpha - \lambda_H$ it must be that $\mu^* = 0$. Hence, $\kappa \geq \alpha - \lambda_H \Rightarrow \mu^* = 0$ and no firms can make use of a referral.

To ensure that this is an equilibrium check that there is no incentive for firms to deviate from their given strategy. There are two cases to consider:

1. $0 < \kappa < \alpha - \lambda_H$: If a firm has a referral opportunity, they have paid the cost κ. If they deviate and hire in the formal market where expected profits are zero, then they will earn negative expected profits. Hence, they have no incentive to deviate. A firm currently not entering the market for referrals can only hire via the formal market, since they have no incumbent worker to enable a referral. From this strategy they will earn zero expected profits. Since $c(\mu^*) = k$ and $c(\cdot)$ is a decreasing function of μ, if a firm were to enter the market for referrals, then the number of firms increases arbitrarily by $\epsilon > 0$. Thus, $\mu^* + \epsilon \;\Rightarrow\; c(\mu^* + \epsilon) < k$. The value of a referral to that additional firm is less than the cost, hence there is no incentive to enter.

2. $\kappa \geq \alpha - \lambda_H$: For values of $\mu \in (0, \infty)$, the horizontal cost curve does not intersect the downward sloping demand curve. All firms are hiring in the formal market, earning zero expected profits. Any firm that deviates and enters would incur non-positive expected profits as the most they could earn is strictly less than $\alpha - \lambda_H$, which less than or equal to the cost of entry κ. Hence, there is no incentive to deviate.

□

Lemma 2

Proof. From proposition 1 when the cost of entry κ is sufficiently low then there is a unique μ which for given parameters $\alpha, \kappa, \lambda_H$ solves

$$\frac{\alpha - \lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)} = \kappa$$

In order to derive equation (8) re-arrange the above equilibrium condition and let μ be a function of α

$$F\left(\alpha, \mu(\alpha)\right) = \alpha - \lambda_H - \kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu(\alpha)}{2(1-\lambda_H)}\right) - \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu(\alpha)}{2\lambda_H}\right) = 0$$

Apply the implicit function theorem around the solution μ^*, where the denominator must not be equal zero

$$\frac{\mathrm{d}\mu^*(\alpha)}{\mathrm{d}\alpha} = -\frac{\frac{\partial F(\alpha,\mu^*(\alpha))}{\partial\alpha}}{\frac{\partial F(\alpha,\mu^*(\alpha))}{\partial\mu^*(\alpha)}}$$

For the denominator, making use of the equations (6) and (7)

$$\begin{aligned}\frac{\partial F\left(\alpha, \mu^*(\alpha)\right)}{\partial\mu} &= -\kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\frac{(1-\alpha)}{2(1-\lambda_H)} - \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu^*)}{2\lambda_H}\right)\frac{\alpha}{2\lambda_H}\\ &= -\kappa\gamma^*\left(\alpha, \lambda_H, \kappa\right)(1-\alpha) - \kappa\phi^*\left(\alpha, \lambda_H, \kappa\right)\alpha\\ &= -\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]\\ &\neq 0\end{aligned}$$

Where, the last line is true, so the implicit function theorem holds, since $\phi, \gamma > 0$. For the numerator, making use of the equations (6) and (7)

$$\begin{aligned}-\frac{\partial F\left(\alpha, \mu^*(\alpha)\right)}{\partial\alpha} &= -\left[1 - \kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\frac{-\mu^*}{2(1-\lambda_H)} - \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu^*)}{2\lambda_H}\right)\frac{\mu^*}{2\lambda_H}\right]\\ &= -\left[1 + \kappa\mu^*\gamma^*\left(\alpha, \lambda_H, \kappa\right) - \kappa\mu^*\phi^*\left(\alpha, \lambda_H, \kappa\right)\right]\\ &= -\left[1 - \kappa\mu^*\left[\phi^* - \gamma^*\right]\right]\end{aligned}$$

The ratio of the two equations derived is

$$\frac{\mathrm{d}\mu^*(\alpha)}{\mathrm{d}\alpha} = \frac{1 - \kappa\mu^*\left[\phi^* - \gamma^*\right]}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]}$$

As required.

For equation (9) also re-arrange the equilibrium condition and let μ be a function of κ

$$F\left(\kappa, \mu(\kappa)\right) = \alpha - \lambda_H - \kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu(\kappa)}{2(1-\lambda_H)}\right) - \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu(\kappa)}{2\lambda_H}\right) = 0$$

Again apply the implicit function theorem around solution μ^* with the normal requirements. The derivative of $F(\kappa, \mu(\kappa))$ with μ is the same as the previous part and so satisfies the requirements for applying the theorem

$$\frac{\partial F(\kappa, \mu^*(\kappa))}{\partial \mu} = -\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]$$

For the derivative of $F(\alpha, \mu(\kappa))$ with κ

$$\begin{aligned}
-\frac{\partial F(\alpha, \mu^*(\kappa))}{\partial \kappa} &= \lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right) \\
&= 2(1-\lambda_H)\gamma^*(\alpha, \lambda_H, \kappa) - 2\lambda_H\phi^*(\alpha, \lambda_H, \kappa) \\
&= 2\left[(1-\lambda_H)\gamma^* + \lambda_H\phi^*\right]
\end{aligned}$$

The ratio of the two equations derived is

$$\frac{\mathrm{d}\mu^*(\kappa)}{\mathrm{d}\kappa} = \frac{-2\left[(1-\lambda_H)\gamma^* + \lambda_H\phi^*\right]}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]}$$

As required.

For equation (10) repeat the same steps as before, except now let μ be a function of λ_H

$$F(\lambda_H, \mu(\lambda_H)) = \alpha - \lambda_H - \kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu(\lambda_H)}{2(1-\lambda_H)}\right) - \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu(\lambda_H)}{2\lambda_H}\right) = 0$$

The derivative of $F(\lambda_H, \mu(\lambda_H))$ with μ is the same as the previous part and so satisfies the requirements for applying the theorem

$$\frac{\partial F(\lambda_H, \mu^*(\lambda_H))}{\partial \mu} = -\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]$$

For the derivative of $F(\lambda_H, \mu(\lambda_H))$ with λ_H

$$\begin{aligned}
-\frac{\partial F(\lambda_H, \mu^*(\lambda_H))}{\partial \lambda_H} &= -1 - \kappa\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + \kappa\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right) \\
&\quad - \kappa\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\frac{\mu^*}{2(1-\lambda_H)^2} + \kappa(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)\frac{\alpha\mu^*}{2\lambda_H^2} \\
&= -1 - \kappa 2\frac{(1-\lambda_H)}{\lambda_H}\gamma^* + \kappa 2\frac{\lambda_H}{1-\lambda_H}\phi^* - \frac{\kappa\mu^*(1-\alpha)}{(1-\lambda_H)}\gamma^* + \frac{\kappa\mu^*\alpha}{\lambda_H}\phi^* \\
&= -1 - \frac{\kappa\left[2(1-\lambda_H)^2\gamma^* - 2\lambda_H^2\phi^* + \mu^*(1-\alpha)\lambda_H\gamma^* - \mu^*\alpha(1-\lambda_H)\phi^*\right]}{\lambda_H(1-\lambda_H)} \\
&= -1 + \frac{\kappa\left[\phi^*\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma^*\left(2(1-\lambda_H)^2 + \lambda_H(1-\alpha)\mu^*\right)\right]}{\lambda_H(1-\lambda_H)}
\end{aligned}$$

The ratio of the two equations derived is

$$\frac{\mathrm{d}\mu^*(\lambda_H)}{\mathrm{d}\lambda_H} = \frac{\left[\phi^*\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma^*\left(2(1-\lambda_H)^2 + \lambda_H(1-\alpha)\mu^*\right)\right]}{\lambda_H(1-\lambda_H)\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} - \frac{1}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]}$$

□

Proposition 3

Proof. Let

$$P(\alpha) = \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right), \qquad Q(\alpha) = \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

Then the derivative of $P(R)$ with respect to α is given by

$$\frac{\partial P_R}{\partial \alpha} = \frac{1}{2}\left[\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}\left[\alpha P(\alpha) + (1-\alpha)Q(\alpha)\right] + \mu^*\left[P(\alpha) - Q(\alpha)\right]\right] \tag{14}$$

Recall that in equilibrium that the following condition holds

$$\frac{\alpha-\lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}=\kappa$$

Hence, as $\kappa \to 0$ then $\mu^* \to \infty$ and $\kappa \to \alpha - \lambda_H$ then $\mu^* \to 0$. The model can be simulated numerically and the sign of equation (14) is increasing and then decreasing, hence the function $P_R(\alpha)$ is single peaked. From the equation in 14 it can be simplified to

$$\alpha \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)+(1-\alpha)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+\frac{\lambda_H}{2(1-\lambda_H)}\exp\left(\frac{-(\alpha-\lambda_H)\mu^*}{2\lambda_H(1-\lambda_H)}\right)\mu^*\kappa \\ -\frac{(1-\lambda_H)}{2\lambda_H}\exp\left(\frac{(\alpha-\lambda_H)\mu^*}{2\lambda_H(1-\lambda_H)}\right)\mu^*\kappa \quad (15)$$

Notice that if $\alpha \simeq \lambda_H$, which is the smallest value α can be then the above reduces to the following

$$\alpha \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)+(1-\alpha)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

which is clearly positive. Hence, $\frac{\partial P_R}{\partial \alpha} > 0$ when α is small and this applies to both parts of the proposition. Let $\alpha = 1$, then the equation (15) becomes

$$\exp\left(-\frac{\mu^*}{2\lambda_H}\right)+\frac{\lambda_H}{2(1-\lambda_H)}\exp\left(\frac{-\mu^*}{2\lambda_H}\right)\mu^*\kappa-\frac{(1-\lambda_H)}{2\lambda_H}\exp\left(\frac{\mu^*}{2\lambda_H}\right)\mu^*\kappa$$

For part 1 let the cost be sufficiently small $\kappa^- = \gamma > 0$ hence this will lead to a large value μ^* . Since the first two expressions of the above equation have the exponential function raised to a negative exponent they will quickly tend to a small number. In particular the second expression, which even though is a product of μ^* and the exponential function, the overall value will be determined by the value of exponential function. The third term is the product of μ^* and the exponential function raised to a positive exponent and so will tend to a large value and dominate the whole expression. Hence, the sign will be negative and $\frac{\partial P_R}{\partial \alpha} < 0$. As the function is single peaked the maximum lies within the interval $\alpha \in (\lambda_H, 1)$. Thus part 1 is proven. For part 2 let the cost be sufficiently large $\kappa^+ \simeq \alpha - \lambda_H$ hence this will lead to a small value $\mu^* \simeq 0$. Hence the above equation the second and third expression become zero as $\exp(0) \times (\alpha - \lambda_H) \times 0 = 0$ and the first expression reduces to 1. Hence, the sign will be positive and $\frac{\partial P_R}{\partial \alpha} > 0$. As the function is single peaked and the slope is positive the maximum is found for $\alpha > 1$.

□

Proposition 4

Proof. In order to show this I will argue that as the proportion of H types λ_H increases that the highest referral wage $\overline{w_R}$ remains unchanged, but that the lowest referral wage (formal market wage) w_M increases. This means that the lower bound of the distribution $F(w_R)$ is increasing as λ_H. Hence given two distributions F, parameterized for λ_H and F', parameterized for λ_H'. Let $\lambda_H < \lambda_H'$. Assume that the market wage for the distribution F' will be greater than that for F e.g. $w_{M'} > w_M$. The probability that a worker under distribution F' when offered the market wage $w_{M'}$ does not have a better alternative offer is zero, $F'(w_{M'}) = 0$, However, it is not zero under the other distribution, $F(w_{M'}) > 0$. Since, F is continuous this argument holds as the referral wage w_R is increased incrementally, until $w_R = \overline{w_R}$. From definition 1 it must be that F' first order stochastically dominates F.

In order to complete the proof, I need to show

$$\frac{\partial\, \overline{w_R}}{\partial \lambda_H}=0, \qquad \frac{\partial w_M}{\partial \lambda_H}>0$$

The highest referral wage is given by equation (4) and the value of a referral c from equation (3)

$$\overline{w_R}=\alpha-\frac{\alpha-\lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}$$

Let

$$\theta = \lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)$$

The derivative of this with respect to λ_H is

$$\begin{aligned}\frac{\partial\theta}{\partial\lambda_H} = \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right) - \exp\left(\frac{\alpha\mu}{2\lambda_H}\right) + \lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)\left[\frac{(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)^2}\right] \\ + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)\left[\frac{\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} - \frac{\alpha\mu^*}{2\lambda_H^2}\right]\end{aligned}$$

Making use of the equations (6) and (7),

$$\begin{aligned}\frac{\partial\theta}{\partial\lambda_H} &= \frac{2(1-\lambda_H)}{\lambda_H}\gamma^* - \frac{2\lambda_H}{(1-\lambda_H)}\phi^* + \gamma^*\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)}{(1-\lambda_H)}\mu^*\right] + \phi^*\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} - \frac{\alpha}{2\lambda_H}\mu^*\right] \\ &= \frac{2(1-\lambda_H)^2\gamma^* - 2\lambda_H^2\phi^* + \mu^*(1-\alpha)\lambda_H)\gamma^* - \mu^*\alpha(1-\lambda_H)\phi^*}{(1-\lambda_H)\lambda_H} + \frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}\left[(1-\alpha)\gamma^* + \alpha\phi^*\right] \\ &= \frac{-\phi\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) + \gamma\left(2(1-\lambda_H)^2 + \lambda_H\alpha\mu^*\right)}{(1-\lambda_H)\lambda_H} + \frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}\left[(1-\alpha)\gamma^* + \alpha\phi^*\right]\end{aligned}$$

Given that

$$\Psi^* = \left[\phi\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma\left(2(1-\lambda_H)^2 + \lambda_H\alpha\mu^*\right)\right]$$

Plugging in the equation for $\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}$ from equation (10) and Ψ^* this further simplifies to

$$\begin{aligned}\frac{\partial\theta}{\partial\lambda_H} &= \frac{-\Psi^*}{(1-\lambda_H)\lambda_H} + \frac{\left[\phi^*\left(2\lambda_H^2 + (1-\lambda_H)\alpha\mu^*\right) - \gamma^*\left(2(1-\lambda_H)^2 + \lambda_H\alpha\mu^*\right)\right]}{\lambda_H(1-\lambda_H)} - \frac{1}{\kappa} \\ &= \frac{-\Psi^*}{(1-\lambda_H)\lambda_H} + \frac{\Psi^*}{\lambda_H(1-\lambda_H)} - \frac{1}{\kappa} \\ &= -\frac{1}{\kappa}\end{aligned}$$

Now taking the derivative of $\overline{w_R}$ using θ and $\frac{\partial\theta}{\partial\lambda_H}$ gives

$$\begin{aligned}\frac{\partial\overline{w_R}}{\partial\lambda_H} &= \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\frac{\partial\theta}{\partial\lambda_H} \\ &= \frac{1}{\theta} - \frac{(\alpha-\lambda_H)}{\theta^2\kappa}\end{aligned}$$

Note that in equilibrium it must be that

$$\kappa = \frac{\alpha-\lambda_H}{\lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}$$

Hence, plugging this into the the partial derivative gives

$$\begin{aligned}\frac{\partial\overline{w_R}}{\partial\lambda_H} &= \frac{1}{\theta} - \frac{(\alpha-\lambda_H)}{\theta^2}\frac{\lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}{\alpha-\lambda_H} \\ &= \frac{1}{\theta} - \frac{1}{\theta^2}\theta \\ &= \frac{1}{\theta} - \frac{1}{\theta} \\ &= 0\end{aligned}$$

The lowest referral wage is given by equation (1)

$$w_M = \frac{\lambda_H\exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)}{\lambda_H\exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)}$$

Note the following derivatives

$$\frac{\partial\left(\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\right)}{\partial \lambda_H} = \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + \lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[-\frac{\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H^2}\right]$$
$$= \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[-\frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right]$$
$$= \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right]$$

Similarly

$$\frac{\partial\left((1-\lambda_H) \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\right)}{\partial \lambda_H} = -\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[-\frac{(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} - \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)^2}\right]$$
$$= -\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[-\frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} - \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]$$
$$= -\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]$$

Let

$$\Gamma = \lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

Hence

$$\frac{\partial \Gamma}{\partial \lambda_H} = \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right] - \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]$$

Also let

$$P(\alpha) = \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right), \qquad Q(\alpha) = \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

This means that the derivative of the lowest referral wage with respect to the proportion of H types is

$$\frac{\partial w_M}{\partial \lambda_H} = \frac{P(\alpha)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right]}{\Gamma} - \frac{\lambda_H P(\alpha)}{\Gamma^2}\frac{\partial \Gamma}{\partial \lambda_H}$$

Plugging in the derivative of Γ gives

$$= \frac{P(\alpha)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right]}{\Gamma} - \left[\frac{\lambda_H P(\alpha)}{\Gamma^2}\right]\left[P(\alpha)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right] - Q(\alpha)\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]\right]$$

$$= \frac{P(\alpha)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right][\lambda_H P(\alpha) + (1-\lambda_H)Q(\alpha)]}{\Gamma^2}$$
$$- \left[\frac{\lambda_H P(\alpha)^2}{\Gamma^2}\right]\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right] + \left[\frac{\lambda_H P(\alpha)Q(\alpha)}{\Gamma^2}\right]\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]$$
$$= \frac{P(\alpha)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right][(1-\lambda_H)Q(\alpha)]}{\Gamma^2} + \left[\frac{\lambda_H P(\alpha)Q(\alpha)}{\Gamma^2}\right]\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]$$
$$= \frac{P(\alpha)Q(\alpha)\left[(1-\lambda_H)\left[1 - \frac{\alpha}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{\alpha\mu^*}{2\lambda_H}\right] + \lambda_H\left[1 + \frac{(1-\alpha)}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H} + \frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right]\right]}{\Gamma^2}$$
$$= \frac{P(\alpha)Q(\alpha)}{\Gamma^2}\left[1 + \frac{(1-\lambda_H)\alpha\mu^*}{2\lambda_H} + \frac{\lambda_H(1-\alpha)\mu^*}{2(1-\lambda_H)} - \frac{1}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}[\alpha - \lambda_H]\right]$$

Given $\frac{P(\alpha)Q(\alpha)}{\Gamma^2} > 0$. If

$$1 + \frac{(1-\lambda_H)\alpha\mu^*}{2\lambda_H} + \frac{\lambda_H(1-\alpha)\mu^*}{2(1-\lambda_H)} - \frac{1}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}[\alpha - \lambda_H] > 0 \Rightarrow \frac{\partial w_M}{\partial \lambda_H} > 0$$

Hence, focusing on the above equation and plugging equation (10) and equilibrium value for κ

$$
\begin{aligned}
&=1+\frac{(1-\lambda_H)\alpha\mu^*}{2\lambda_H}+\frac{\lambda_H(1-\alpha)\mu^*}{2(1-\lambda_H)}-\frac{1}{2}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}[\alpha-\lambda_H] \\
&=1+\frac{1}{2}\left[\frac{(1-\lambda_H)\alpha\mu^*}{\lambda_H}+\frac{\lambda_H(1-\alpha)\mu^*}{(1-\lambda_H)}-\frac{\Psi^*[\alpha-\lambda_H]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}+\frac{[\alpha-\lambda_H]}{\kappa[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&=1+\frac{1}{2}\left[\frac{(1-\lambda_H)\alpha\mu^*}{\lambda_H}+\frac{\lambda_H(1-\alpha)\mu^*}{(1-\lambda_H)}-\frac{\Psi^*[\alpha-\lambda_H]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}+\frac{\lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}{[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&=1+\frac{1}{2}\left[\frac{\left[(1-\lambda_H)^2\alpha+\lambda_H^2(1-\alpha)\right]\mu^*[\alpha\phi^*+(1-\alpha)\gamma^*]-\Psi^*[\alpha-\lambda_H]+2\lambda_H(1-\lambda_H)^2\gamma^*+2(1-\lambda_H)\lambda_H^2\phi^*}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right]
\end{aligned}
$$

Plug in the value of Ψ^* to give

$$
\begin{aligned}
&=1+\frac{1}{2}\left[\frac{\left[1\alpha-2\lambda_H\alpha+\lambda_H^2\right]\mu^*[\alpha\phi^*+(1-\alpha)\gamma^*]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&-\frac{1}{2}\left[\frac{\left[\phi\left(2\lambda_H^2+(1-\lambda_H)\alpha\mu^*\right)-\gamma^*\left(2(1-\lambda_H)^2+\lambda_H(1-\alpha)\mu^*\right)\right][\alpha-\lambda_H]-2\lambda_H(1-\lambda_H)^2\gamma^*-2(1-\lambda_H)\lambda_H^2\phi^*}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=1+\frac{1}{2}\left[\frac{\phi^*\left[\alpha^2(1-2\lambda_H)\mu^*+\lambda_H^2\alpha\mu^*-2\lambda_H^2(\alpha-\lambda_H)-(1-\lambda_H)\alpha(\alpha-\lambda_H)\mu^*\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&\quad+\frac{1}{2}\left[\frac{\gamma^*\left[\alpha(1-\alpha)(1-2\lambda_H)\mu^*+\lambda_H^2(1-\alpha)\mu^*+2(1-\lambda_H)^2(\alpha-\lambda_H)+\lambda_H(1-\alpha)\mu^*(\alpha-\lambda_H)\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&\qquad+\frac{1}{2}\left[\frac{2\lambda_H(1-\lambda_H)^2\gamma^*+2(1-\lambda_H)\lambda_H^2\phi^*}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=1+\frac{1}{2}\left[\frac{\phi^*\left[\alpha^2(1-2\lambda_H)\mu^*+\lambda_H^2\alpha\mu^*-2\lambda_H^2\alpha+2\lambda_H^3-(1-\lambda_H)\alpha^2\mu^*+(1-\lambda_H)\lambda_H\alpha\mu^*+2\lambda_H^2-2\lambda_H^3\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&+\frac{1}{2}\left[\frac{\gamma^*\left[\alpha(1-\alpha)(1-2\lambda_H)\mu^*+\lambda_H^2(1-\alpha)\mu^*+2(1-\lambda_H)^2(\alpha-\lambda_H)+\lambda_H(1-\alpha)\mu^*(\alpha-\lambda_H)+2\lambda_H(1-\lambda_H)^2\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=1+\frac{1}{2}\left[\frac{\phi^*\left[\alpha^2(1-2\lambda_H)\mu^*+\lambda_H^2\alpha\mu^*-2\lambda_H^2\alpha-(1-\lambda_H)\alpha^2\mu^*+(1-\lambda_H)\lambda_H\alpha\mu^*+2\lambda_H^2\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&\quad+\frac{1}{2}\left[\frac{\gamma^*\left[\alpha(1-\alpha)(1-2\lambda_H)\mu^*+\lambda_H^2(1-\alpha)\mu^*+2(1-\lambda_H)^2\alpha+\lambda_H(1-\alpha)\mu^*(\alpha-\lambda_H)\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=1+\frac{1}{2}\left[\frac{\phi^*\left[2\lambda_H^2(1-\alpha)+\alpha\mu^*\left[\alpha(1-2\lambda_H)+\lambda_H^2-(1-\lambda_H)\alpha+(1-\lambda_H)\lambda_H\right]\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&\qquad+\frac{1}{2}\left[\frac{\gamma^*\left[2(1-\lambda_H)^2\alpha+(1-\alpha)\mu^*\left[\alpha(1-2\lambda_H)+\lambda_H^2+\lambda_H\alpha-\lambda_H^2\right]\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&=1+\frac{1}{2}\left[\frac{\phi^*\left[2\lambda_H^2(1-\alpha)+\alpha\mu^*(1-\alpha)\lambda_H\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}+\frac{\gamma^*\left[2(1-\lambda_H)^2\alpha+(1-\alpha)\mu^*\alpha(1-\lambda_H)\right]}{\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]}\right] \\
&=1+\frac{\left[\phi^*\left[2\lambda_H^2(1-\alpha)+\alpha\mu^*(1-\alpha)\lambda_H\right]+\gamma^*\left[2(1-\lambda_H)^2\alpha+(1-\alpha)\mu^*\alpha(1-\lambda_H)\right]\right]}{2\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]} \\
&=1+\frac{\left[(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)[2\lambda_H(1-\alpha)+\alpha\mu^*(1-\alpha)]+\lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)[2(1-\lambda_H)\alpha+(1-\alpha)\mu^*\alpha(1-\lambda_H)]\right]}{4\lambda_H(1-\lambda_H)[\alpha\phi^*+(1-\alpha)\gamma^*]} \\
&>0,\qquad\forall\lambda_H,\alpha,\kappa
\end{aligned}
$$

□

Proposition 5

Proof. First consider the homphily parameter α. I will show that increases in α will decrease the lowest referral wage (formal market wage) w_M, but will increase referral wage $\overline{w_R}$. This means that the lower bound of the distribution $F(w_R)$ is decreasing in α, but the upper bound of the distribution is increasing in α. Given two distributions F, parameterized for α and F', parameterized for α'. Let $\alpha < \alpha'$. Also let w_M and $\overline{w_R}$ be the lower and upper bounds of distribution F. Then $F'(w_M) > F(w_M)$ but $F'(\overline{w_R}) < F(\overline{w_R})$. Which violates the conditions for first order stochastic dominance.

In order to complete the proof, I need to show

$$\frac{\partial\ \overline{w_R}}{\partial \alpha} > 0, \qquad \frac{\partial w_M}{\partial \alpha} < 0$$

The lowest referral wage is given by equation (1)

$$w_M = \frac{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)}$$

Note the following derivatives

$$\begin{aligned}\frac{\partial\left(\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\right)}{\partial\alpha} &= \lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[-\frac{\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} - \frac{\mu^*}{2\lambda_H}\right] \\ &= -\frac{\exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)}{2}\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} + \mu^*\right]\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial\left((1-\lambda_H)exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\right)}{\partial\alpha} &= (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[-\frac{(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} + \frac{\mu^*}{2(1-\lambda_H)}\right] \\ &= -\frac{\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)}{2}\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} - \mu^*\right]\end{aligned}$$

Let

$$\Gamma = \lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

Hence

$$\frac{\partial\Gamma}{\partial\alpha} = -\frac{1}{2}\left[\exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} + \mu^*\right] + \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} - \mu^*\right]\right]$$

Also let

$$P(\alpha) = \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right), \qquad Q(\alpha) = \exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)$$

This means that the derivative of the lowest referral wage with respect to homophily is

$$\begin{aligned}
\frac{\partial w_M}{\partial \alpha} &= -\frac{P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]}{2\Gamma} - \frac{\lambda_H P(\alpha)}{\Gamma^2}\frac{\partial \Gamma}{\partial \alpha} \\
&= \frac{-P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]}{2\Gamma} + \left[\frac{\lambda_H P(\alpha)}{2\Gamma^2}\right]\left[P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]+Q(\alpha)\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]\right] \\
&= \frac{-P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right][\lambda_H P(\alpha)+(1-\lambda_H)Q(\alpha)]}{2\Gamma^2} + \left[\frac{\lambda_H P(\alpha)}{2\Gamma^2}\right]\left[P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]+Q(\alpha)\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]\right] \\
&= \frac{-(1-\lambda_H)Q(\alpha)P(\alpha)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]+\lambda_H P(\alpha)Q(\alpha)\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]}{2\Gamma^2} \\
&= \frac{Q(\alpha)P(\alpha)\left[\lambda_H\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]-(1-\lambda_H)\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]\right]}{2\Gamma^2} \\
&= \frac{Q(\alpha)P(\alpha)\left[\lambda_H(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\lambda_H\mu^*-(1-\lambda_H)\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-(1-\lambda_H)\mu^*\right]}{2\Gamma^2} \\
&= \frac{Q(\alpha)P(\alpha)}{2\Gamma^2}\left[\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]
\end{aligned}$$

Given $\dfrac{P(\alpha)Q(\alpha)}{2\Gamma^2} > 0$. If

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^* < 0 \Rightarrow \frac{\partial w_M}{\partial \alpha} < 0$$

Hence, focusing on the above equation and plugging equation (8) and equilibrium value for κ

$$\begin{aligned}
&= \frac{1}{\kappa\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}(\lambda_H-\alpha) - \frac{\mu^*\left(\phi^*-\gamma^*\right)}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}(\lambda_H-\alpha)-\mu^* \\
&= \frac{-\lambda_H\exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)-(1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]} - \frac{\mu^*\left(\phi^*-\gamma^*\right)(\lambda_H-\alpha)}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}-\mu^* \\
&= -\left[\frac{2(1-\lambda_H)\gamma^*+2\lambda_H\phi^*+\mu^*\left(\phi^*-\gamma^*\right)(\lambda_H-\alpha)+\mu^*\alpha\phi^*+\mu^*(1-\alpha)\gamma^*}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}\right] \\
&= -\left[\frac{2(1-\lambda_H)\gamma^*+2\lambda_H\phi^*+\mu^*\phi^*\lambda_H-\mu^*\alpha\phi^*-\mu^*\gamma^*\lambda_H+\mu^*\alpha\gamma^*+\mu^*\alpha\phi^*+\mu^*\gamma^*-\mu^*\alpha\gamma^*}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}\right] \\
&= -\left[\frac{2(1-\lambda_H)\gamma^*+2\lambda_H\phi^*+\mu^*\phi^*\lambda_H+\mu^*\gamma^*(1-\lambda_H)}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}\right] \\
&= -\left[\frac{(1-\lambda_H)\gamma^*(2+\mu^*)+\lambda_H\phi^*(2+\mu^*)}{\left[\alpha\phi^*+(1-\alpha)\gamma^*\right]}\right] \\
&< 0, \qquad \forall \lambda_H, \alpha, \kappa
\end{aligned}$$

The highest referral wage is given by equation (4) and the value of a referral c from equation (3)

$$\overline{w_R} = \alpha - \frac{\alpha-\lambda_H}{\lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}$$

Let

$$\theta = \lambda_H\exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)+(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)$$

The derivative of this with respect to α is

$$\frac{\partial\theta}{\partial\alpha} = \lambda_H\exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)\left[\frac{(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\frac{\mu^*}{2(1-\lambda_H)}\right]+(1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)\left[\frac{\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\frac{\mu^*}{2\lambda_H}\right]$$

Making use of the equations (6) and (7),

$$\frac{\partial\theta}{\partial\alpha} = \gamma^*\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}-\mu^*\right]+\phi^*\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha}+\mu^*\right]$$

Hence, the derivative of $\overline{w_R}$ with respect to α is

$$\begin{aligned}
\frac{\partial \overline{w_R}}{\partial \alpha} &= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\gamma^*\left[(1-\alpha)\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} - \mu^*\right] + \phi^*\left[\alpha\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} + \mu^*\right]\right] \\
&= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]\frac{\mathrm{d}\mu^*}{\mathrm{d}\alpha} + \mu^*(\phi^*-\gamma^*)\right] \\
&= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\frac{\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} - \frac{\mu^*\left(\phi^*-\gamma^*\right)\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]}{\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} + \mu^*(\phi^*-\gamma^*)\right] \\
&= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\frac{1}{\kappa} - \mu^*\left(\phi^*-\gamma^*\right) + \mu^*(\phi^*-\gamma^*)\right] \\
&= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\frac{1}{\kappa}\right] \\
&= 1 - \frac{1}{\theta} + \frac{(\alpha-\lambda_H)}{\theta^2}\left[\frac{\theta}{(\alpha-\lambda_H)}\right] \\
&= 1 - -\frac{1}{\theta} + \frac{1}{\theta} \\
&= 1 > 0, \qquad \forall \lambda_H, \alpha, \kappa
\end{aligned}$$

By the same reasoning the distribution F with respect to changes in κ violates first order stochastic dominance. Given two distributions F, parameterized for κ and F', parameterized for κ'. Let $\kappa < \kappa'$. Also let w_M and $\overline{w_R}$ be the lower and upper bounds of distribution F. Then $F'(w_M) > F(w_M)$ but $F'(\overline{w_R}) < F(\overline{w_R})$. Which violates the conditions for first order stochastic dominance.

In order to complete the proof, I need to show

$$\frac{\partial\, \overline{w_R}}{\partial \kappa} < 0, \qquad \frac{\partial w_M}{\partial \kappa} > 0$$

Again the lowest referral wage is given by equation (1)

$$w_M = \frac{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)}{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)}$$

Notice that only μ^* is a function of κ so the derivative is simply

$$\begin{aligned}
\frac{\partial w_M}{\partial \kappa} &= \frac{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\frac{-\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa}}{\lambda_H \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)\frac{-\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa} + (1-\lambda_H)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\frac{-(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa}} \\
&= \frac{\alpha \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right)}{\alpha \exp\left(-\frac{\alpha\mu^*}{2\lambda_H}\right) + (1-\alpha)\exp\left(-\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)} \\
&> 0, \qquad \forall \lambda_H, \alpha, \kappa
\end{aligned}$$

The highest referral wage is given by equation (4) and the value of a referral c from equation (3)

$$\overline{w_R} = \alpha - \frac{\alpha - \lambda_H}{\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)}$$

Let

$$\theta = \lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)$$

The derivative of this with respect to α is

$$\frac{\partial \theta}{\partial \kappa} = \lambda_H \exp\left(\frac{(1-\alpha)\mu}{2(1-\lambda_H)}\right)\left[\frac{(1-\alpha)}{2(1-\lambda_H)}\frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa}\right] + (1-\lambda_H)\exp\left(\frac{\alpha\mu}{2\lambda_H}\right)\left[\frac{\alpha}{2\lambda_H}\frac{\mathrm{d}\mu^*}{\mathrm{d}\lambda_H}\right]$$

Making use of the equations (6) and (7),

$$\frac{\partial \theta}{\partial \kappa} = [(1-\alpha)\gamma^* + \alpha\phi^*] \frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa}$$

Hence, the derivative of $\overline{w_R}$ with respect to α is

$$\begin{aligned}
\frac{\partial \overline{w_R}}{\partial \kappa} &= \frac{(\alpha-\lambda_H)}{\theta^2}[(1-\alpha)\gamma^* + \alpha\phi^*]\frac{\mathrm{d}\mu^*}{\mathrm{d}\kappa} \\
&= \frac{(\alpha-\lambda_H)}{\theta^2}[(1-\alpha)\gamma^* + \alpha\phi^*]\frac{-2\left[(1-\lambda_H)\gamma^* + \lambda_H\phi^*\right]}{\kappa\left[\alpha\phi^* + (1-\alpha)\gamma^*\right]} \\
&= \frac{(\alpha-\lambda_H)}{\theta^2}\frac{-2\left[(1-\lambda_H)\gamma^* + \lambda_H\phi^*\right]}{\kappa} \\
&= -\frac{2\left[(1-\lambda_H)\gamma^* + \lambda_H\phi^*\right]}{\theta} \\
&= -\frac{\left[(1-\lambda_H)\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right)\frac{2}{2(1-\lambda_H)} + \lambda_H(1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)\frac{2}{2\lambda_H}\right]}{\theta} \\
&= -\frac{\left[\lambda_H \exp\left(\frac{(1-\alpha)\mu^*}{2(1-\lambda_H)}\right) + (1-\lambda_H)\exp\left(\frac{\alpha\mu^*}{2\lambda_H}\right)\right]}{\theta} \\
&= -\frac{\theta}{\theta} \\
&= -1 < 0, \qquad \forall \lambda_H, \alpha, \kappa
\end{aligned}$$

□

Proposition 6

Proof. The only additional step to the proof for proposition 1 is to check that a firm which hires an L type worker has no incentive to deviate and make use of a referral. Note that due to homophily the probability of firm with an incumbent L worker connects with an unemployed H worker is less than the formal market $1-\hat{\alpha} < lambda_H$. Suppose, that a firm who hired an L type in period 1 deviates and offers a referral wage. The expected profit of such as firm is given by

$$\mathbb{E}\Pi_2^L(\text{ref}) = (1-\hat{\alpha})P(H \text{ accept})(1-w_R) - \hat{\alpha}P(L \text{ accept})w_R$$

Since, a firm is negligible the workers will not have their beliefs altered. This means that $P(H \text{ accept})$ is the same as given in equation (2). As before in any equilibrium there will be some distribution of referral wages offered, where at any wage the expected payoff must be the same. Plugging in w_M gives the following equation

$$\begin{aligned}
c^L = (1-\hat{\alpha})\exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right) & \frac{(1-\lambda_H)\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)}{\lambda_H \exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)} \\
& - \hat{\alpha}\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)\frac{\lambda_H \exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right)}{\lambda_H \exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)}
\end{aligned}$$

Where, the main difference to the firm with an H incumbent is that the probability of connecting with an unemployed H type is switched. Hence, some manipulation will yield

$$c^L = \frac{\left[(1-\lambda_H)(1-\hat{\alpha}) - \lambda_H\hat{\alpha}\right]\exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right)\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)}{\lambda_H \exp\left(-\frac{\hat{\alpha}\mu}{2\lambda_H}\right) + (1-\lambda_H)\exp\left(-\frac{(1-\hat{\alpha})\mu}{2(1-\lambda_H)}\right)}$$

Now if $(1-\lambda_H)(1-\hat{\alpha}) - \lambda_H\hat{\alpha} > 0$ then there is an incentive for the firm to deviate. But

$$\begin{aligned}
0 <& (1-\lambda_H)(1-\hat{\alpha}) - \lambda_H\hat{\alpha} \\
=& (1-\hat{\alpha}) - \lambda_H(1-\hat{\alpha}) - \lambda_H\hat{\alpha} \\
=& 1-\hat{\alpha} - \lambda_H + \hat{\alpha}\lambda_H - \lambda_H\hat{\alpha} \\
=& 1-\hat{\alpha} - \lambda_H
\end{aligned}$$

This implies that $1 - \hat{\alpha} > \lambda_H$ a contradiction. Hence, there is no incentive to deviate.

□